

KMS WEIGHTS ON GRAPH C^* -ALGEBRAS

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ABSTRACT. The paper contains a study of the gauge invariant KMS weights for a generalized gauge action on a graph C^* -algebra. When the graph is irreducible and has the property that there are at most countably many roads to infinity in the graph, a complete description is given of the structure of KMS weights for the gauge action. The structure is very rich and is identical with the structure of KMS states for the restriction of the action to any corner defined by a projection in the fixed point algebra.

1. INTRODUCTION

In 1980 Bratteli, Elliott and Kishimoto proved a remarkable theorem concerning the structure of inverse temperatures and the corresponding simplexes of KMS states for a one-parameter group of automorphisms on a unital C^* -algebra, [BEK]. Their result says that if a given structure of simplexes can be realized inside a metrizable compact convex set in such a way that the closure properties which the KMS states have inside the state space of a unital C^* -algebra are satisfied, then the structure is in fact the structure of KMS states of a periodic one-parameter group of automorphisms acting on a unital and simple separable C^* -algebra. In particular, it follows that any KMS structure which occurs with some unital separable C^* -algebra, can also be realized with a unital and simple separable C^* -algebra. This fact is in striking contrast to the observation that in practically all cases where it has been possible to determine the structure of inverse temperatures and simplexes of KMS states of an a priori given one-parameter action on a simple C^* -algebra, the structure has been disappointingly poor; often with only one possible inverse temperature and a unique KMS state. The gauge action on graph C^* -algebras is no exception if one sticks with finite graphs, [EFW], but it is the purpose of the present paper to show how radically this changes when infinite graphs are considered.

We extend first the study of KMS weights on graph algebras which was initiated in [Th1] by allowing the graph to have sinks and infinite emitters. In most of the paper we work in the same generality as in [Th1], dealing with generalized gauge actions, but in this introduction where only some of the results are described, attention is restricted to the gauge action on the C^* -algebra of a strongly connected graph G . There are then no sinks to consider, but there may be plenty of infinite emitters. As in [Th1] the KMS weights are given by regular Borel measures on the path space of the graph, which besides the infinite paths now also contains finite paths terminating at an infinite emitter. This division of the path space leads to a similar division of the KMS weights, depending on the supports of the corresponding measures. If the measure is supported on the finite paths, we say that the KMS weight is a *boundary*

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KMS weight and if the finite paths is a null set for the measure, that it is a *harmonic KMS weight*.

In order to have any KMS weights at all the adjacency matrix of the graph must have 'finite powers of all orders'. This means that for a given vertex v and a given natural number n the number $a(n)$ of paths of length n from v back to itself must be finite. In fact, the exponential growth rate of $a(n)$ must be finite. The logarithm of this growth rate is the *Gurevich entropy* $h(G)$ of the graph and there are no KMS weights when the Gurevich entropy is infinite, and when it is finite there are no β -KMS weights when $\beta < h(G)$. The graphs for which this paper describes the structure of KMS weights completely are those with at most countably many exits. Here an *exit* is a tail-equivalence class of *exit paths*, and an exit path is a sequence $(t_i)_{i=1}^{\infty}$ of vertexes in the graph such that there is an edge from t_i to t_{i+1} for all i and such that t_i goes to infinity in the natural sense. Each exit contributes an interval¹ of inverse temperatures in $[h(G), \infty)$, and for each β in the interval there is an extremal ray of β -KMS weights, uniquely determined by the condition that the corresponding measures are supported on the exit. It is these KMS weights that are responsible for the rich structure of the KMS weights that may be realized with graphs of this kind, because it turns out that the intervals of inverse temperatures which the exits contribute are independent and can be almost arbitrary. To formulate this more precisely, we have to distinguish between when G is recurrent and when it is transient, [Ru]. In terms of the numbers $a(n)$ introduced above, G is transient when the sum

$$\sum_{n=1}^{\infty} a(n)e^{-nh(G)}$$

is finite and recurrent when it is not. Furthermore, we have to distinguish between graphs that are row-finite, in the sense that the out-degree at every vertex is finite, and those that are not row-finite. Concerning the latter class of graphs we obtain the following theorems.

Theorem 1.1. *Let $N \in \{1, 2, 3, \dots\} \cup \{\infty\}$ and let $h \in]0, \infty[$ be a positive real number. Let \mathbb{I} be a finite or countably infinite collection of intervals in $]h, \infty[$.*

There is a strongly connected recurrent graph G with Gurevich entropy $h(G) = h$, such that the set of exits in G is in bijective correspondence with \mathbb{I} , and for $\beta \geq h$ there are the following extremal β -KMS weights for the gauge action on $C^(G)$:*

- *For $\beta > h$ there are N extremal rays of boundary β -KMS weights, in bijective correspondence with the infinite emitters in G , and the rays of extremal harmonic β -KMS weights are in bijective correspondence with the set*

$$\{I \in \mathbb{I} : \beta \in I\}.$$

- *For $\beta = h$ there are no boundary h -KMS weights and a unique ray of extremal harmonic h -KMS weights.*

Theorem 1.2. *Let $N \in \{1, 2, 3, \dots\} \cup \{\infty\}$ and let $h \in]0, \infty[$ be a positive real number. Let \mathbb{I} be a finite or countably infinite collection of intervals in $[h, \infty[$.*

¹Here and in the rest of the paper an interval is to be understood in the broadest sense possible. It may be empty, open, closed, half-open or even degenerate, i.e. consist of a single number only.

There is a strongly connected transient graph G with Gurevich entropy $h(G) = h$, such that the set of exits in G is in bijective correspondence with \mathbb{I} , and for $\beta \geq h$ there are the following rays of extremal β -KMS weights for the gauge action on $C^(G)$: There are N extremal rays of boundary β -KMS weights, in bijective correspondence with the infinite emitters in G , and the rays of extremal harmonic β -KMS weights are in bijective correspondence with the set $\{I \in \mathbb{I} : \beta \in I\}$.*

Before we get to the construction of the graphs mentioned in the two theorems, we obtain results demonstrating that the structures described are the most general that can be obtained from strongly connected graphs with infinite emitters and at most countably many exits.

The row-finite case, where there are no infinite emitters, must be handled separately because the results from [Th1] show that there are β -KMS weights for all $\beta > h(G)$ when G is not finite and hence the total freedom in the choice of intervals in the theorems above does not persist to the row-finite case. We show that in the row-finite case there must be at least one exit which contributes the maximal possible interval of inverse temperatures, namely $]h(G), \infty[$ in the recurrent case and $[h(G), \infty[$ in the transient case. Once this restriction is identified it is not difficult to modify the construction from the case with infinite emitters and show that it is also the only restriction. In this way we obtain results, formulated in Theorem 7.1 and Theorem 7.2 below, which describe the possibilities in the row-finite case.

The proofs of the main results are constructive in a very literal sense. They present a small collection of building blocks and describe methods to choose and put together a finite or countable collection of such building blocks to form a strongly connected graph for which the gauge action has any desired KMS spectrum, restricted only by the general limitations which graphs with at most countably many exits are subject to.

By restricting attention to corners in $C^*(G)$ the above structure of KMS weights becomes the structure of KMS states on a unital and simple C^* -algebra: A vertex v in G determines a projection 1_v in $C^*(G)$ in a canonical way, and the gauge action restricts to the corner $1_v C^*(G) 1_v$ in $C^*(G)$. There is then a bijective correspondence between the rays of KMS weights on $C^*(G)$ and the KMS states on $1_v C^*(G) 1_v$, and the results of the paper describe therefore the structure of KMS states that can occur on such a corner when the graph has at most countably many exits.

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2. KMS WEIGHTS, MEASURES AND ALMOST HARMONIC VECTORS

Recall, [KV], [Th1], that a weight ψ on the C^* -algebra A is *proper* when it is non-zero, densely defined and lower semi-continuous. For such a weight, set

$$\mathcal{N}_\psi = \{a \in A : \psi(a^*a) < \infty\}.$$

Let $\alpha : \mathbb{R} \rightarrow \text{Aut } A$ be a point-wise norm-continuous one-parameter group of automorphisms on A . Let $\beta \in \mathbb{R}$. Following [C] we say that a proper weight ψ on A is a β -KMS weight for α when

- i) $\psi \circ \alpha_t = \psi$ for all $t \in \mathbb{R}$, and

- ii) for every pair $a, b \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^*$ there is a continuous and bounded function F defined on the closed strip D_β in \mathbb{C} consisting of the numbers $z \in \mathbb{C}$ whose imaginary part lies between 0 and β , and is holomorphic in the interior of the strip and satisfies that

$$F(t) = \psi(a\alpha_t(b)), \quad F(t + i\beta) = \psi(\alpha_t(b)a)$$

for all $t \in \mathbb{R}$.²

A β -KMS weight ψ with the property that

$$\sup \{\psi(a) : 0 \leq a \leq 1\} = 1$$

will be called a β -KMS state. This is consistent with the standard definition of KMS states, [BR], except when $\beta = 0$ in which case our definition requires also that a 0-KMS state, which is a trace state, is α -invariant. A β -KMS weight is *extremal* when every β -KMS weight φ such that $\varphi \leq \psi$ has the form $\varphi = \lambda\psi$ for some $\lambda \in]0, 1]$.

2.1. The étale groupoid of a countable graph. Let G be a countable directed graph with vertex set V and edge set E . For an edge $e \in E$ we denote by $s(e) \in V$ its source and by $r(e) \in V$ its range. An *infinite path* in G is an element $p \in E^{\mathbb{N}}$ such that $r(p_i) = s(p_{i+1})$ for all i . A finite path $p = p_1 p_2 \dots p_n$ is defined similarly. The number of edges in p is its *length* and we denote it by $|p|$. A vertex $v \in V$ will be considered as a finite path of length 0.

We let $P(G)$ denote the set of infinite paths in G and $P_f(G)$ the set of finite paths in G . We extend the source map to $P(G)$ such that $s(p) = s(p_1)$ when $p = (p_i)_{i=1}^\infty$, and the range and source maps to $P_f(G)$ such that $s(p) = s(p_1)$ and $r(p) = r(p_n)$ when $|p| = n \geq 1$, and $s(v) = r(v) = v$ when $v \in V$.

A vertex v which does not emit any edge is a *sink*, while a vertex v which emits infinitely many edges will be called an *infinite emitter*. The union V_∞ of sinks and infinite emitters will play a crucial role in the following.

The C^* -algebra $C^*(G)$ of the graph G is the universal C^* -algebra generated by a collection $S_e, e \in E$, of partial isometries and a collection $P_v, v \in V$, of mutually orthogonal projections subject to the conditions that

- 1) $S_e^* S_e = P_{r(e)}, \forall e \in E$,
- 2) $S_e S_e^* \leq P_{s(e)}, \forall e \in E$,
- 3) $P_v \geq \sum_{e \in s^{-1}(v)} S_e S_e^*, \forall v \in V$, and
- 4) $P_v = \sum_{e \in s^{-1}(v)} S_e S_e^*, \forall v \in V \setminus V_\infty$.

It will be crucial for our approach to the graph C^* -algebra $C^*(G)$ that it can be realized as the (reduced) C^* -algebra $C_r^*(\mathcal{G})$ of an étale groupoid \mathcal{G} through the construction introduced by J. Renault in [Re]. The relevant groupoid \mathcal{G} was constructed by A. Paterson in [Pa]. See in particular Corollary 3.9 in [Pa].

As a set the unit space Ω_G of the groupoid \mathcal{G} is the union

$$\Omega_G = P(G) \cup Q(G),$$

where

$$Q(G) = \{p \in P_f(G) : r(p) \in V_\infty\}$$

²As in [Th1] we apply the definition from [C] for the action α_{-t} in order to use the same sign convention as in [BR], for example.

is the set of finite paths that terminate at a vertex in V_∞ . Note that $Q(G)$ is countable and that $V_\infty \subseteq Q(G)$. For any $p \in P_f(G)$, $|p| \geq 1$, set

$$Z(p) = \{q \in \Omega_G : |q| \geq |p|, q_i = p_i, i = 1, 2, \dots, |p|\},$$

and

$$Z(v) = \{q \in \Omega_G : s(q) = v\}$$

when $v \in V$. When $\nu \in P_f(G)$ and F is a finite subset of $P_f(G)$, set

$$Z_F(\nu) = Z(\nu) \setminus \left(\bigcup_{\mu \in F} Z(\mu) \right). \quad (2.1)$$

The sets $Z_F(\nu)$ form a basis of compact and open subsets for a locally compact Hausdorff topology on Ω_G . When $\mu \in P_f(G)$ and $x \in \Omega_G$, we can define the concatenation $\mu x \in \Omega_G$ in the obvious way when $r(\mu) = s(x)$. The groupoid \mathcal{G} consists of the elements in $\Omega_G \times \mathbb{Z} \times \Omega_G$ of the form

$$(\mu x, |\mu| - |\mu'|, \mu' x),$$

for some $x \in \Omega_G$ and some $\mu, \mu' \in P_f(G)$. The product in \mathcal{G} is defined by

$$(\mu x, |\mu| - |\mu'|, \mu' x)(\nu y, |\nu| - |\nu'|, \nu' y) = (\mu x, |\mu| + |\nu| - |\mu'| - |\nu'|, \nu' y),$$

when $\mu' x = \nu y$, and the involution by $(\mu x, |\mu| - |\mu'|, \mu' x)^{-1} = (\mu' x, |\mu'| - |\mu|, \mu x)$. To describe the topology on \mathcal{G} , let $Z_F(\mu)$ and $Z_{F'}(\mu')$ be two sets of the form (2.1) with $r(\mu) = r(\mu')$. The topology we shall consider has as a countable basis the sets of the form

$$\{(\mu x, |\mu| - |\mu'|, \mu' x) : \mu x \in Z_F(\mu), \mu' x \in Z_{F'}(\mu')\}. \quad (2.2)$$

With this topology \mathcal{G} becomes an étale locally compact Hausdorff groupoid and we can consider the reduced C^* -algebra $C_r^*(\mathcal{G})$ as in [Re]. As shown by Paterson in [Pa] there is an isomorphism $C^*(G) \rightarrow C_r^*(\mathcal{G})$ which sends S_e to 1_e , where 1_e is the characteristic function of the compact and open set

$$\{(ex, 1, r(e)x) : x \in \Omega_G\} \subseteq \mathcal{G},$$

and P_v to 1_v , where 1_v is the characteristic function of the compact and open set

$$\{(vx, 0, vx) : x \in \Omega_G\} \subseteq \mathcal{G}.$$

In the following we use the identification $C^*(G) = C_r^*(\mathcal{G})$ and identify Ω_G with the unit space of \mathcal{G} via the embedding $\Omega_G \ni x \mapsto (x, 0, x)$.

Remark 2.1. The set $P(G)$ is usually considered as a metric space, e.g. with the metric

$$d(p, q) = \sum_{i=1}^{\infty} 2^{-i} (1 - \delta(p_i, q_i))$$

where $\delta(e, e) = 1$ and $\delta(e, f) = 0$ when $e \neq f$. It is easy to see that the topology on $P(G)$ defined by such a metric is the same as the topology which $P(G)$ inherits as a subset of Ω_G . In particular, the two topologies define the same Borel subsets in $P(G)$.

2.2. Generalized gauge actions on $C^*(G)$ and their gauge invariant KMS weights. Let $F : E \rightarrow \mathbb{R}$ be a function. We extend F to a function $F : P_f(G) \rightarrow \mathbb{R}$ such that

$$F(p_1 p_2 \cdots p_n) = \sum_{i=1}^n F(p_i)$$

when $p = p_1 p_2 \cdots p_n$ is a path of length $n \geq 1$ in G , and $F(v) = 0$ when $v \in V$. We can then define a continuous function $c_F : \mathcal{G} \rightarrow \mathbb{R}$ such that

$$c_F(ux, |u| - |u'|, u'x) = F(u) - F(u').$$

Since c_F is a continuous homomorphism it gives rise to a continuous one-parameter automorphism group α^F on $C_r^*(\mathcal{G})$ defined such that

$$\alpha_t^F(f)(\gamma) = e^{itc_F(\gamma)} f(\gamma)$$

when $f \in C_c(\mathcal{G})$, cf. [Re]. When F is constant 1 this action is known as *the gauge action* on $C^*(G)$.

Let $\beta \in \mathbb{R}$. Following the terminology used in [Th1] we say that a regular Borel measure m on Ω_G is (\mathcal{G}, c_F) -conformal with exponent β when

$$m(s(W)) = \int_{r(W)} e^{\beta c_F(r_W^{-1}(x))} dm(x) \quad (2.3)$$

for every open bi-section $W \subseteq \mathcal{G}$. Here r_W^{-1} denotes the inverse of $r : W \rightarrow r(W)$. When the function F is fixed we shall often in the following refer to a (\mathcal{G}, c_F) -conformal measure with exponent β as a β -KMS measure. The connection to β -KMS weights is given by the following theorem which is a special case of Theorem 2.2 in [Th1].

Theorem 2.2. *There is a bijective correspondence $m \mapsto \varphi_m$ between the non-zero (\mathcal{G}, c_F) -conformal measures m with exponent β and the gauge invariant β -KMS weights φ_m for the action α^F on $C^*(G)$. The bijection is such that*

$$\varphi_m(f) = \int_{\Omega_G} f(z) dm(z)$$

when $f \in C_c(\mathcal{G})$.

In terms of the canonical generators, the β -KMS weight φ_m is given by

$$\varphi_m(S_e S_f^*) = \delta(e, f) e^{-\beta F(e)} m(Z(r(e))) \text{ and } \varphi_m(P_v) = m(Z(v)).$$

Remark 2.3. To obtain the theorem above we apply Theorem 2.2 in [Th1] with $c = c_F$ and c_0 equal to the homomorphism that gives the gauge action, i.e. $c_0 = c_F$ with $F = 1$. However, when F is strictly positive everywhere or strictly negative everywhere we can apply Theorem 2.2 in [Th1] with $c = c_0 = c_F$ instead and obtain in that case a version of Theorem 2.2 with the words 'gauge invariant' deleted. This shows that when F is either strictly positive or strictly negative everywhere, all KMS weights for α^F are gauge invariant. The same is true whenever $C^*(G)$ is simple by Proposition 5.6 in [CT2]. In all these cases the results we obtain about gauge invariant KMS weights therefore hold with the words 'gauge invariant' deleted. But in general when zero is not a forbidden value for F , or F is allowed to change sign, there can be KMS weights and states that are not gauge invariant. See [N] and [CT1].

Before we restrict the attention entirely to KMS weights rather than states, we want to point out that when $C^*(G)$ is simple there is a bijection from rays of KMS weights on $C^*(G)$ onto the KMS states of some of its corners. By a *ray* of KMS weights we mean here a set of the form $\{\lambda\psi : \lambda > 0\}$ for some KMS weight ψ .

Theorem 2.4. *Let $\alpha : \mathbb{R} \rightarrow \text{Aut } A$ be a point-wise norm-continuous one-parameter group of automorphisms on a C^* -algebra A . Let p be a projection in the fixed point algebra of α such that p is full in A . For all $\beta \in \mathbb{R}$ the map*

$$\psi \mapsto \psi(p)^{-1}\psi|_{pAp}$$

is a bijection between the set of rays of β -KMS weights for α and the β -KMS states for the restriction of α to pAp .

Proof. It follows from Theorem 3.2 in [LN] that a β -KMS state for the restriction of α to pAp has a unique extension to a β -KMS weight on A , cf. Remark 3.3 (i) in [LN]. Hence all we need to do is to prove that the map in the statement is well-defined, and for this it suffices to show that $0 < \psi(p) < \infty$ when ψ is a β -KMS weight for α . It was shown in Lemma 3.1 of [CT2] that $\psi(p) < \infty$. Assume for a contradiction that $\psi(p) = 0$. It follows from Proposition 2.5.22 in [BR] that A contains a dense $*$ -algebra \mathcal{A} consisting of analytic elements for α . Let $a, b \in \mathcal{A}$ and set $a' = \alpha_{\frac{i\beta}{2}}(a)$. Then $ap = \alpha_{-\frac{i\beta}{2}}(a'p)$ and hence

$$\psi(apbb^*pa^*) \leq \|b\|^2\psi(ap a^*) = \|b\|^2\psi\left(\alpha_{-\frac{i\beta}{2}}(a'p)\alpha_{-\frac{i\beta}{2}}(a'p)^*\right).$$

By using the alternative formulation of the KMS condition given in Proposition 1.11 in [KV], we find that

$$\psi\left(\alpha_{-\frac{i\beta}{2}}(a'p)\alpha_{-\frac{i\beta}{2}}(a'p)^*\right) = \psi(pa'^*a'p) \leq \|a'\|^2\psi(p) = 0.$$

Thus $\psi(apbb^*pa^*) = 0$. By taking $b = p$ it follows in particular that $pa^* \in \mathcal{N}_\psi$. If we instead take $a = b^*$ and $b = p$, it follows that $pb \in \mathcal{N}_\psi$, and we conclude that $apb = (pa^*)^*pb \in \mathcal{N}_\psi^*\mathcal{N}_\psi$. Now recall that ψ extends to a positive linear functional on the $*$ -algebra $\mathcal{N}_\psi^*\mathcal{N}_\psi$, cf. page 842 in [KV]. We can therefore use the Cauchy-Schwarz inequality to conclude that $\psi(xx^*) = 0$ for all x that are linear combinations of elements apb with $a, b \in \mathcal{A}$. That p is full means that elements of the form apb span a dense subspace of $C^*(G)$, and from what we have just shown it follows that $\psi(xx^*) = 0$ for all elements x in this subspace. The lower semi-continuity of ψ implies then that $\psi = 0$; a contradiction. Hence $\psi(p) > 0$ as required. \square

For the actions on $C^*(G)$ we consider here, a natural choice of projection to which Theorem 2.4 applies is one of the projections P_v . This leads to the following

Corollary 2.5. *Let $C^*(G)$ be a simple graph C^* -algebra and v a vertex in G . Let P_v be the corresponding projection. The map*

$$\psi \mapsto \psi(P_v)^{-1}\psi|_{P_v C^*(G) P_v}$$

is a bijection from the rays of β -KMS weights for α^F on $C^(G)$ onto the β -KMS states for the restriction of α^F to $P_v C^*(G) P_v$.*

2.3. KMS measures and super-harmonic functions. Given the function $F : E \rightarrow \mathbb{R}$ and a real number $\beta \in \mathbb{R}$ we define the matrix $A(\beta) = (A(\beta)_{uw})$ over V by

$$A(\beta)_{uw} = \sum_{\{e \in E : s(e)=u, r(e)=w\}} e^{-\beta F(e)}.$$

When $\beta = 0$ the matrix $A = A(0)$ is the *adjacency matrix* of G , i.e.

$$A_{vw} = \# \{e \in E : s(e) = v, r(e) = w\}. \quad (2.4)$$

Note that $A(\beta)_{uw}$ can be infinite, i.e. $A(\beta)_{uw} \in [0, \infty]$. Nonetheless we can define the powers $A(\beta)^n$ of $A(\beta)$ in the usual recursive way:

$$A(\beta)_{uw}^n = \sum_{v \in V} A(\beta)_{uv} A(\beta)_{vw}^{n-1},$$

where we use the convention that $0 \cdot \infty = \infty \cdot 0 = 0$. We define $A(\beta)^0$ to be the identity matrix, i.e. $A(\beta)_{uu}^0 = 1$ when $u = w$ and $A(\beta)_{uw}^0 = 0$ when $u \neq w$. We shall use the matrix $A(\beta)$ to count the weighted paths between vertexes in G . For this note that for each $n \in \mathbb{N}$,

$$A(\beta)_{vw}^n = \sum_{\mu} e^{-\beta F(\mu)}$$

where the sum is over all paths μ of length n from v to w .

Given a β -KMS measure m on Ω_G we define ψ_v , $v \in V$, such that

$$\psi_v = m(Z(v)). \quad (2.5)$$

Note that $\psi_v < \infty$ since m is regular and $Z(v)$ is compact in Ω_G .

Lemma 2.6. *Let m be a β -KMS measure on Ω_G . The vector $\psi_v = m(Z(v))$, $v \in V$, has the following two properties:*

- 1) $\sum_{w \in V} A(\beta)_{vw} \psi_w \leq \psi_v$, $v \in V$, and
- 2) $\sum_{w \in V} A(\beta)_{vw} \psi_w = \psi_v$, $v \in V \setminus V_\infty$.

The identity 2) holds for all $v \in V$ when $m(Q(G)) = 0$.

Proof. Consider a vertex $v \in V$ and an edge $e \in s^{-1}(v)$. Then $\{(ex, 1, x) : x \in Z(r(e))\}$ is an open bi-section in \mathcal{G} . As m is (\mathcal{G}, c_F) -conformal with exponent β this implies that

$$m(Z(r(e))) = e^{\beta F(e)} m(Z(e)).$$

Assume first that $v \in V_\infty$. Then $Z(v)$ is a disjoint union $Z(v) = \{v\} \cup \bigcup_{e \in s^{-1}(v)} Z(e)$, and hence

$$\begin{aligned} \psi_v &= m(\{v\}) + \sum_{e \in s^{-1}(v)} m(Z(e)) = m(\{v\}) + \sum_{e \in s^{-1}(v)} e^{-\beta F(e)} m(Z(r(e))) \\ &= m(\{v\}) + \sum_{w \in V} A(\beta)_{vw} m(Z(w)) \geq \sum_{w \in V} A(\beta)_{vw} \psi_w. \end{aligned}$$

This shows that 1) holds when $v \in V_\infty$. When $v \in V \setminus V_\infty$ or $m(\{v\}) = 0$ the term $m(\{v\})$ does not enter and we obtain 2) instead. \square

It follows from 1) in Lemma 2.6 by induction that

$$\sum_{w \in V} A(\beta)_{vw}^n \psi_w \leq \psi_v \quad (2.6)$$

for all $n \in \mathbb{N}$ and all $v \in V$. In the following we shall say that ψ is $A(\beta)$ -harmonic when

$$\sum_{w \in V} A(\beta)_{vw} \psi_w = \psi_v$$

for all $v \in V$, and that ψ is *almost $A(\beta)$ -harmonic* when conditions 1) and 2) in Lemma 2.6 both hold. This terminology is inspired by the notion of harmonic and super-harmonic functions used in the theory of Markov chains, cf. [Wo].

We aim to prove the following theorem.

Theorem 2.7. *The map $m \mapsto \psi$ given by (2.5) is a bijection from β -KMS measures on Ω_G onto the set of almost $A(\beta)$ -harmonic vectors on V .*

The injectivity of the map $m \mapsto \psi$ is a consequence of the following lemmas.

Lemma 2.8. *Let m be a β -KMS measure on Ω_G . Then*

$$m(Z(\mu)) = e^{-\beta F(\mu)} m(Z(r(\mu))) \quad (2.7)$$

for all $\mu \in P_f(G)$.

Proof. This follows from (2.3) applied to the open bisection

$$W = \{(\mu x, |\mu|, r(\mu)x) : x \in Z(r(\mu))\}.$$

□

Lemma 2.9. *Let $v \in V$ be a vertex and let m, m' be finite Borel measures on $Z(v)$. Assume that $m(Z(\mu)) = m'(Z(\mu))$ for every finite path $\mu \in P_f(G)$ with $s(\mu) = v$. Then $m = m'$.*

Proof. The intersection of a finite collection of cylinder sets $Z(\mu)$ is again a cylinder set or empty. These cylinder sets therefore form a π -system in the sense of [Co] and the conclusion follows from Corollary 1.6.2 in [Co] since the σ -algebra generated by the cylinder sets contains all open sets and hence the Borel σ -algebra. □

Proof of injectivity in Theorem 2.7: Let m and m' be β -KMS measures on Ω_G such that $m(Z(v)) = m'(Z(v))$ for every vertex $v \in V$. Fix a vertex $v \in V$ and note that $m(Z(v)) = m'(Z(v)) < \infty$ by regularity of the two measures. It follows from Lemma 2.8 that $m(Z(\mu)) = m'(Z(\mu))$ and then from Lemma 2.9 that m and m' agree on all Borel subsets of $Z(v)$. Since $v \in V$ was arbitrary and $Z(v), v \in V$, is a countable Borel partition of Ω_G it follows that the two measures are identical. □

For the proof of the surjectivity part in Theorem 2.7 we shall use the following lemma which is an extension of the Riesz decomposition used in connection with countable state Markov chains, e.g. Lemma 4.2 in [Sa]. The only difference is that we here do not require that the matrix is stochastic or sub-stochastic, but the proof is the same. In the present setting the details of the proof are presented in [Th3].

Lemma 2.10. *Let $\psi = (\psi_v)_{v \in V} \in [0, \infty]^V$ be a non-negative vector such that*

$$\sum_{w \in V} A(\beta)_{vw} \psi_w \leq \psi_v$$

for all $v \in V$. It follows that there are unique non-negative vectors $h, k \in [0, \infty]^V$ such that h is $A(\beta)$ -harmonic and

$$\psi_v = h_v + \sum_{w \in V} \sum_{n=0}^{\infty} A(\beta)_{vw}^n k_w \quad (2.8)$$

for all $v \in V$. The vector k is given by

$$k_v = \psi_v - \sum_{w \in V} A(\beta)_{vw} \psi_w, \quad v \in V.$$

To prove the surjectivity of the map in Theorem 2.7 we shall consider the two components in the decomposition (2.8) of an almost $A(\beta)$ -harmonic vector separately. One virtue of this approach is that it shows how the decomposition of an almost $A(\beta)$ -harmonic vector given by Lemma 2.10 corresponds to the decomposition of Ω_G as the disjoint union of $P(G)$ and $Q(G)$.

3. BOUNDARY AND HARMONIC KMS MEASURES

Recall that a subset $A \subseteq \Omega_G$ is \mathcal{G} -invariant when $r(s^{-1}(A)) \subseteq A$ and $s(r^{-1}(A)) \subseteq A$.

Lemma 3.1. *Let m be (\mathcal{G}, c_F) -conformal with exponent β and let $A \subseteq \Omega_G$ be a \mathcal{G} -invariant Borel subset. Then the Borel measure m_A , given by*

$$m_A(B) = m(A \cap B),$$

is (\mathcal{G}, c_F) -conformal with exponent β .

Proof. Let W be an open bi-section in \mathcal{G} . Since m is (\mathcal{G}, c_F) -conformal with exponent β , the two Borel measures on W ,

$$B \mapsto m(s(B))$$

and

$$B \mapsto \int_{r(B)} e^{\beta c_F(r_W^{-1}(x))} dm(x),$$

agree on open sets. Note that they are both regular by Proposition 7.2.3 in [Co] and therefore equal since they agree on open sets. It follows that

$$m_A(s(W)) = m(s(W \cap s^{-1}(A))) = \int_{r(W \cap s^{-1}(A))} e^{\beta c_F(r_W^{-1}(x))} dm(x).$$

Since $r(W \cap s^{-1}(A)) = r(W) \cap A$ because A is \mathcal{G} -invariant,

$$\begin{aligned} & \int_{r(W \cap s^{-1}(A))} e^{\beta c_F(r_W^{-1}(x))} dm(x) \\ &= \int_{r(W) \cap A} e^{\beta c_F(r_W^{-1}(x))} dm(x) = \int_{r(W)} e^{\beta c_F(r_W^{-1}(x))} dm_A(x). \end{aligned}$$

□

Lemma 3.2. *Let m be a β -KMS measure on Ω_G . There are unique β -KMS measures m_1 and m_2 such that $m = m_1 + m_2$, and $m_1(Q(G)) = m_2(P(G)) = 0$.*

Proof. This follows from Lemma 3.1 since $P(G)$ and $Q(G)$ are \mathcal{G} -invariant in Ω_G . □

The decomposition in Lemma 3.2 is a version for weights of the decomposition of KMS states into finite and infinite type used by Carlsen and Larsen in [CL]. Since 'finite' and 'infinite' have other meanings in connection with measures and weights we prefer to alter the terminology. We will say that a β -KMS measure m is a *boundary β -KMS measure* when $m(P(G)) = 0$ and a *harmonic β -KMS measure* when $m(Q(G)) = 0$. This terminology is justified (or so the author hopes) by the

fact that Ω_G in many cases is a completion of $P(G)$ with $Q(G)$ as the boundary, and by the description of the harmonic KMS measures we obtain in the following.

From now on we will tacitly identify a harmonic β -KMS measure with its restriction to $P(G)$. The β -KMS weight defined by a non-zero boundary β -KMS measure or a non-zero harmonic β -KMS measure will be called a *boundary β -KMS weight* and a *harmonic β -KMS weight*, respectively.

3.1. Boundary KMS measures. Let $\beta \in \mathbb{R}$. For any vertex $v \in V_\infty$ we can consider the Borel measure

$$m_v = \sum_{\{u \in Q(G) : r(u)=v\}} e^{-\beta F(u)} \delta_u$$

on Ω_G where δ_u denotes the Dirac measure at u . Consider an open bisection W in \mathcal{G} . Then $T = r \circ s^{-1} : s(W) \cap Q(G) \rightarrow r(W) \cap Q(G)$ is a bijection determined by the condition that $(T(u), |T(u)| - |u|, u) \in W$. Since $c_F(T(u), |T(u)| - |u|, u) = F(T(u)) - F(u)$ we find that

$$\begin{aligned} m_v(s(W)) &= \sum_{\{u \in s(W) \cap Q(G) : r(u)=v\}} e^{-\beta F(u)} \\ &= \sum_{\{u \in s(W) \cap Q(G) : r(u)=v\}} e^{\beta(F(T(u)) - F(u))} e^{-\beta F(T(u))} \\ &= \sum_{\{u \in s(W) \cap Q(G) : r(u)=v\}} e^{\beta(c_F(r_W^{-1}(T(u))))} e^{-\beta F(T(u))} = \int_{r(W)} e^{\beta c_F(r_W^{-1}(x))} dm_v(x). \end{aligned}$$

This shows that m_v satisfies condition (2.3), and hence is a β -KMS measure if and only if it is regular. We say that a vertex $v \in V_\infty$ is β -summable when

$$\sum_{n=0}^{\infty} A(\beta)_{wv}^n < \infty$$

for all $w \in V$.

Lemma 3.3. *Let $v \in V_\infty$. The Borel measure m_v is regular, and hence a β -KMS measure if and only if v is β -summable.*

Proof. m_v is regular if and only if $m_v(K) < \infty$ for every compact subset $K \subseteq \Omega_G$, cf. e.g. Proposition 7.2.3 in [Co]. Since $Z(w)$, $w \in V$, is a cover of Ω_G by open and compact subsets it follows that m_v is regular if and only if $m_v(Z(w)) < \infty$ for all $w \in V$. The lemma follows therefore from the observation that

$$m_v(Z(w)) = \sum_{\{u \in Q(G) : s(u)=w, r(u)=v\}} e^{-\beta F(u)} = \sum_{n=0}^{\infty} A(\beta)_{wv}^n. \quad (3.1)$$

□

Theorem 3.4. *Let S be a set of β -summable vertexes in V_∞ and $t \in]0, \infty[^S$ a vector such that*

$$\sum_{v \in S} \sum_{n=0}^{\infty} A(\beta)_{wv}^n t_v < \infty \quad (3.2)$$

for all $w \in V$. Then

$$m = \sum_{v \in S} t_v m_v \quad (3.3)$$

is a boundary β -KMS measure, and every non-zero boundary β -KMS measure arises in this way.

Proof. Note that m satisfies (2.3) since each m_v does. The condition (3.2) is therefore exactly what is needed to ensure that m is regular and hence a boundary β -KMS measure. It remains therefore only to prove that every non-zero boundary β -KMS measure m arises like this. Consider an element $u \in Q(G)$ and set $v = r(u) \in V_\infty$. Assume first that v is an infinite emitter. Let $F_1 \subseteq F_2 \subseteq F_3 \subseteq \dots$ be finite subsets of edges such that $s^{-1}(v) = \bigcup_i F_i$. For each i

$$W_i = \{(ux, |u|, vx) : x \in Z_{F_i}(v)\}$$

is an open bisection in \mathcal{G} such that $s(W_i) = Z_{F_i}(v)$ and $r(W_i) = Z_{uF_i}(u)$, where

$$uF_i = \{ux : x \in F_i\}.$$

Note that $\bigcap_i Z_{F_i}(v) = \{v\}$, $\bigcap_i Z_{uF_i}(u) = \{u\}$ and $c_F(ux, |u|, vx) = F(u)$. It follows therefore from (2.3) that

$$\begin{aligned} m(\{v\}) &= \lim_{i \rightarrow \infty} m(Z_{F_i}(v)) = \lim_{i \rightarrow \infty} \int_{r(W_i)} e^{\beta c_F(r_{W_i}^{-1}(z))} dm(z) \\ &= \lim_{i \rightarrow \infty} \int_{r(W_i)} e^{\beta F(u)} dm(z) = e^{\beta F(u)} m(\{u\}). \end{aligned} \quad (3.4)$$

If instead v is a sink, the point $(u, |u|, v)$ is isolated in \mathcal{G} , and it is an open bisection in itself. It follows therefore from (2.3) that

$$m(\{v\}) = e^{\beta F(u)} m(\{u\}),$$

also when v is a sink. For each $w \in V$,

$$\begin{aligned} \sum_{n=0}^{\infty} A(\beta)_{wv}^n m(\{v\}) &= \sum_{\{u \in Q(G) \cap Z(w) : r(u)=v\}} e^{-\beta F(u)} m(\{v\}) \\ &= \sum_{\{u \in Q(G) \cap Z(w) : r(u)=v\}} m(\{u\}) \leq m(Z(w)) < \infty, \end{aligned} \quad (3.5)$$

by regularity of m . Hence v is β -summable whenever $m(\{v\}) > 0$. Set $S = \{v \in V_\infty : m(\{v\}) \neq 0\}$. By assumption m is supported on the countable set $Q(G)$ and hence

$$m = \sum_{u \in Q(G)} m(\{u\}) \delta_u = \sum_{v \in S} \sum_{\{u \in Q(G) : r(u)=v\}} m(\{v\}) e^{-\beta F(u)} \delta_u = \sum_{v \in S} m(\{v\}) m_v.$$

This shows that (3.3) holds when we set $t_v = m(\{v\})$. To show that also (3.2) holds, note that the sets $\{u \in Q(G) \cap Z(w) : r(u) = v\}$, $v \in S$, are mutually disjoint subsets of $Z(w)$. Using (3.5) this implies that

$$\sum_{v \in S} \sum_{n=0}^{\infty} A(\beta)_{wv}^n m(\{v\}) = \sum_{v \in S} \sum_{\{u \in Q(G) \cap Z(w) : r(u)=v\}} m(\{u\}) \leq m(Z(w)) < \infty.$$

□

The decomposition (3.3) is unique since $m_v(\{v'\}) = 0$ when $v' \neq v$, $v, v' \in V_\infty$. Theorem 3.4 therefore has the following

Corollary 3.5. *The map $v \mapsto m_v$ gives a bijective correspondence from the set of β -summable vertexes in V_∞ onto the rays of non-zero extremal boundary β -KMS measures.*

The corresponding result for states is Corollary 5.18(1) and Proposition 5.8 in [CL].

3.2. Harmonic KMS measures. Let m be a harmonic β -KMS measure on Ω_G and consider the vector ψ given by (2.5). It follows from Lemma 2.6 that ψ is $A(\beta)$ -harmonic. To show that all $A(\beta)$ -harmonic vectors arise this way it is convenient to relate our setup to the theory of countable state Markov chains through the following device.

Let ψ be a non-zero $A(\beta)$ -harmonic vector. Set

$$E_0 = \{e \in E : \psi_{r(e)} \neq 0\}.$$

We define a matrix $B(\beta)$ over E_0 by

$$B(\beta)_{e_0, e_1} = \begin{cases} \psi_{r(e_0)}^{-1} \psi_{r(e_1)} e^{-\beta F(e_1)}, & \text{when } r(e_0) = s(e_1), \\ 0, & \text{otherwise.} \end{cases} \quad (3.6)$$

Then

$$\sum_{e_1 \in E_0} B(\beta)_{e_0, e_1} = \psi_{r(e_0)}^{-1} \sum_{e_1 \in E_0, s(e_1)=r(e_0)} e^{-\beta F(e_1)} \psi_{r(e_1)} = \psi_{r(e_0)}^{-1} \sum_{w \in V} A(\beta)_{r(e_0), w} \psi_w = 1$$

since ψ is $A(\beta)$ -harmonic. Thus $B(\beta)$ is stochastic and gives therefore rise to a Markov chain with E_0 as state space for any choice of initial distribution on E_0 , cf. Theorem 1.12 (b) in [Wo]. When we fix an edge $e \in E_0$ and take as the initial distribution the Dirac measure at e we get in this way a probability measure \Pr_e on $E_0^\mathbb{N}$ defined on the σ -algebra \mathcal{A} generated by the cylinder sets in $E_0^\mathbb{N}$ by Theorem 1.12 (a) in [Wo]. When $(a_i)_{i=0}^n \in E_0^{n+1}$, the corresponding cylinder set in $E_0^\mathbb{N}$ is

$$C((a_i)_{i=0}^n) = \{(x_i)_{i=0}^\infty \in E_0^\mathbb{N} : x_i = a_i, i = 0, 1, 2, \dots, n\}. \quad (3.7)$$

By definition

$$\Pr_e(C((a_i)_{i=0}^n)) = \psi_{r(e)}^{-1} \psi_{r(a_n)} e^{-\beta \sum_{i=1}^n F(a_i)} \quad (3.8)$$

when $a_0 = e$ and $r(a_i) = s(a_{i+1})$, $i = 0, 1, 2, \dots, n-1$, while $\Pr_e(C((a_i)_{i=0}^n)) = 0$ in all other cases.

In the following we set $Z'(\mu) = Z(\mu) \cap P(G)$ when $\mu \in P_f(G)$.

Lemma 3.6. *$B \cap E_0^\mathbb{N} \in \mathcal{A}$ for every Borel subset $B \subseteq Z'(e)$, and*

$$B \mapsto \Pr_e(B \cap E_0^\mathbb{N}) \quad (3.9)$$

is a Borel probability measure on $Z'(e)$.

Proof. The subsets A of $Z'(e)$ such that $A \cap E_0^\mathbb{N} \in \mathcal{A}$ form a σ -algebra and the cylinders $Z'(\mu)$ generate the Borel σ -algebra in $Z'(e)$ so it suffices to show that $Z'(\mu) \cap E_0^\mathbb{N} \in \mathcal{A}$ when μ is a finite path in G with e as initial edge. Fix such a path μ . When $\nu = e_0 e_1 \dots e_n \in P_f(G)$ we set

$$C(\nu) = \{(x_i)_{i=0}^\infty \in E_0^\mathbb{N} : x_i = e_i, i = 0, 1, \dots, n\}.$$

Then $C(\nu) = \emptyset$ when $e_i \notin E_0$ for some i , and otherwise $C(\nu)$ is the cylinder set $C((e_i)_{i=0}^n)$. In any case $C(\nu) \in \mathcal{A}$. Let $P_n(\mu)$ denote the set of finite paths ν in G of length $n \geq |\mu|$ such that $Z(\nu) \subseteq Z(\mu)$. Then

$$Z'(\mu) \cap E_0^{\mathbb{N}} = \bigcap_{k > |\mu|} \left(\bigcup_{\nu \in P_k(\mu)} C(\nu) \right) \in \mathcal{A},$$

completing the proof of the first assertion. For the second note that $E_0^{\mathbb{N}} \setminus Z'(e)$ is contained in a countable union of cylinder sets (3.7) for each of which either $a_0 \neq e$ or $r(a_i) \neq s(a_{i+1})$ for some $i = 0, 1, 2, \dots, n-1$. Such sets are null-sets for Pr_e by definition. Hence $\text{Pr}_e(Z'(e)) = \text{Pr}_e(E_0^{\mathbb{N}}) = 1$. \square

It can happen that $Z'(e) \not\subseteq E_0^{\mathbb{N}}$, but it follows from Lemma 3.6 that $\text{Pr}_e(E_0^{\mathbb{N}} \setminus Z'(e)) = 0$ which justifies that we in the following use the notation Pr_e also for the measure on $Z'(e)$ given by (3.9).

Lemma 3.7. *Let ψ be an $A(\beta)$ -harmonic vector. There is a harmonic β -KMS measure m_ψ on Ω_G such that $\psi_v = m_\psi(Z(v))$ for all $v \in V$.*

Proof. When $\psi = 0$, set $m_\psi = 0$. Assume then that $\psi \neq 0$. Let $e \in E_0$ and let Pr_e be the Borel probability measure on $Z'(e)$ from Lemma 3.6. For every vertex $v \in V$ we define a Borel measure m^v on $Z(v)$ such that $m^v = 0$ when $s^{-1}(v) \cap E_0 = \emptyset$ and

$$m^v(B) = \sum_{e \in s^{-1}(v) \cap E_0} e^{-\beta F(e)} \psi_{r(e)} \text{Pr}_e(B \cap Z'(e)),$$

when $s^{-1}(v) \cap E_0 \neq \emptyset$. Then

$$m^v(Z(v)) = \sum_{e \in s^{-1}(v) \cap E_0} e^{-\beta F(e)} \psi_{r(e)} = \sum_{w \in V} A(\beta)_{vw} \psi_w = \psi_v.$$

When we define m_ψ such that

$$m_\psi(B) = \sum_{v \in V} m^v(B \cap Z(v))$$

we have therefore obtained a Borel measure on Ω_G such that $m_\psi(Z(v)) = m_\psi(Z'(v)) = \psi_v$ for all $v \in V$. In particular, m_ψ is finite on compact subsets of Ω_G because such a set is covered by finitely many $Z(v)$'s. Hence m_ψ is a regular Borel measure by Proposition 7.2.3 in [Co]. Note that m_ψ is supported on $P(G)$. What remains is now only to show that (2.3) holds when W is an open bisection in \mathcal{G} . For this note first that from the definition of m_ψ and (3.8) it follows that

$$m_\psi(Z(\mu)) = m_\psi(Z'(\mu)) = e^{-\beta F(\mu)} \psi_{r(\mu)} \quad (3.10)$$

for all $\mu \in P_f(G)$. Next observe that W is the union of countably many sets of the form (2.2). Note that the set (2.2) is a Borel subset of $\{(\mu x, |\mu| - |\mu'|, \mu'x) : x \in \Omega_G\}$. We can therefore write W as a countable disjoint union $W = \bigcup_i W_i$ of Borel subsets W_i such that there are finite paths $\mu_i, \mu'_i \in P_f(G)$ with $r(\mu_i) = r(\mu'_i)$ and a subset $B_i \subseteq \Omega_G$ with

$$W_i = \{(\mu_i x, |\mu_i| - |\mu'_i|, \mu'_i x) : x \in B_i\}.$$

Note that B_i is a Borel subset of Ω_G because $r : W \rightarrow r(W)$ is a homeomorphism. Since the W_i 's are mutually disjoint, to establish (2.3) it suffices to fix i and show that

$$m_\psi(s(W_i)) = \int_{r(W_i)} e^{\beta c_F(r_W^{-1}(z))} dm_\psi(z). \quad (3.11)$$

Note that $s(W_i) = \{\mu'_i x : x \in B_i\}$ and $r(W_i) = \{\mu_i x : x \in B_i\}$. Let $v = r(\mu_i) = r(\mu'_i)$ and consider the two finite Borel measures on $Z(v)$ given by

$$B \mapsto \int_{\{\mu_i x : x \in B\}} e^{\beta c_F(r_W^{-1}(z))} dm_\psi(z)$$

and

$$B \mapsto m_\psi(\{\mu'_i x : x \in B\}).$$

To show that they are equal it suffices therefore by Lemma 2.9 to check that they agree on cylinder sets $Z(\mu)$ with $s(\mu) = v$. For this note that c_F is constant equal to $F(\mu_i) - F(\mu'_i)$ on W_i . By using (3.10) it follows that

$$\begin{aligned} \int_{\{\mu_i x : x \in Z(\mu)\}} e^{\beta c_F(r_W^{-1}(z))} dm_\psi(z) &= e^{\beta(F(\mu_i) - F(\mu'_i))} m_\psi(\{\mu_i x : x \in Z'(\mu)\}) \\ &= e^{\beta(F(\mu_i) - F(\mu'_i))} e^{-\beta F(\mu_i \mu)} \psi_{r(\mu)} = e^{-\beta F(\mu'_i \mu)} \psi_{r(\mu)} = m_\psi(\{\mu'_i x : x \in Z'(\mu)\}). \end{aligned} \quad (3.12)$$

Thus m_ψ is a harmonic β -KMS measure and the proof is complete. \square

Theorem 3.8. *The map $m \mapsto \psi$ given by (2.5) is a bijection from the harmonic β -KMS measures m on Ω_G onto the $A(\beta)$ -harmonic vectors for $A(\beta)$.*

Proof. Surjectivity is Lemma 3.7 and injectivity was proved after Lemma 2.9. \square

Proof of surjectivity in Theorem 2.7: Let ψ be an almost $A(\beta)$ -harmonic vector. We must find a β -KMS measure m such that $m(Z(u)) = \psi_u$ for all $u \in V$. To this end let h, k be the vectors arising from Lemma 2.10. Note that k is supported on V_∞ since ψ is almost $A(\beta)$ -harmonic. It follows that

$$\sum_{v \in V_\infty} \sum_{n=0}^{\infty} A(\beta)_{wv}^n k_v = \psi_w - h_w \leq \psi_w < \infty.$$

and then $m_2 = \sum_{v \in V_\infty} k_v m_v$ is a boundary β -KMS measure by Theorem 3.4. It follows from Theorem 3.8 that there is a harmonic β -KMS measure m_1 such that $m_1(Z(v)) = h_v$. Then $m = m_1 + m_2$ is a β -KMS measure, and by combining (3.1) with (2.8) we find that

$$m(Z(u)) = h_u + \sum_{v \in V_\infty} k_v m_v(Z(u)) = h_u + \sum_{v \in V_\infty} \sum_{n=0}^{\infty} A(\beta)_{wv}^n k_v = \psi_u. \quad \square$$

To sum up, by combining Theorem 2.2 and Theorem 2.7 we get the following corollary which will be the basis for the following investigations.

Corollary 3.9. *There are affine bijections between the following three sets.*

- The gauge invariant β -KMS weights for α^F .
- The non-zero β -KMS measures on Ω_G .
- The non-zero almost $A(\beta)$ -harmonic vectors on V .

Equally important is the division of the KMS weights which arise from these bijections into harmonic KMS weights and boundary KMS weights.

4. KMS WEIGHTS ON SIMPLE GRAPH C^* -ALGEBRAS

Recall that a subset $H \subseteq V$ is *hereditary* when $e \in E$, $s(e) \in H \Rightarrow r(e) \in H$ and *saturated* when

$$v \in V \setminus V_\infty, r(s^{-1}(v)) \subseteq H \Rightarrow v \in H.$$

In the following we say that G is *cofinal* when the only non-empty subset of V which is both hereditary and saturated is V itself. This condition is fulfilled when $C^*(G)$ is simple, and it is a result of Szymanski that the converse is almost also true, cf. Theorem 12 in [Sz]. We note that when G is *strongly connected*, meaning that for every pair of vertexes v, w there is a finite path μ such that $s(\mu) = v$ and $r(\mu) = w$, then V contains no proper non-empty hereditary subset and G is therefore also cofinal.

Lemma 4.1. *Let G be cofinal and $H \subseteq V$ a non-empty hereditary subset. Set $H_0 = H$ and define $H_i, i \geq 1$, such that*

$$H_i = H_{i-1} \cup \{v \in V \setminus V_\infty : r(s^{-1}(v)) \subseteq H_{i-1}\}.$$

Then $\bigcup_{i=0}^\infty H_i = V$.

Proof. The union is hereditary and saturated. □

Since a sink is a hereditary subset of V , Lemma 4.1 has the following

Corollary 4.2. *Let G be a cofinal graph. Then*

- 1) *G contains at most one sink.*
- 2) *If $P(G) \neq \emptyset$ there is no sink in G .*
- 3) *If V contains a sink, there is for each $u \in V$ a natural number N such that $A(\beta)_{uw}^n = 0$ for all $w \in V$ when $n \geq N$.*

As in [Th1] we consider the set NW_G of non-wandering vertexes. These are the vertexes v for which there is a finite path (a loop) $\mu \in P_f(G)$ such that $|\mu| \geq 1$ and $s(\mu) = r(\mu) = v$. When G is cofinal the vertexes in NW_G and the edges $e \in E$ with $s(e) \in NW_G$ form a sub-graph of G which we shall also denote by NW_G .

Proposition 4.3. *Assume that G is cofinal and that $NW_G \neq \emptyset$. The sub-graph NW_G is strongly connected and the vertexes in NW_G form a hereditary subset of V with the following property: For every $v \in V$ there is an $N_v \in \mathbb{N}$ such that $r(\nu) \in NW_G$ for all $\nu \in P_f(G)$ with $|\nu| \geq N_v$ and $s(\nu) = v$.*

Proof. Consider a path $e_1 e_2 \cdots e_n \in P_f(G)$ such that $n \geq 1$ and $s(e_1) = r(e_n)$, and let L be the set of vertexes v with the property that $v = s(\mu)$ for some $\mu \in P_f(G)$ with $r(\mu) \in \bigcup_{i=1}^n \{r(e_i)\}$. Then $V \setminus L$ is both hereditary and saturated, and it is not all of V since $s(e_1) \in L$, so it is empty. This has several consequence. It follows in particular that we can go back and forth between loops in G and hence that NW_G is strongly connected. It also implies that NW_G is hereditary. For this it suffices to show that $r(e) \in NW_G$ when $s(e) = s(e_1)$. But since $r(e) \in L$ there is a path from $r(e)$ to $s(e)$ and $r(e)$ is therefore contained in a loop, i.e. $r(e) \in NW_G$. Finally, to establish the last statement we apply Lemma 4.1 with $H = NW_G$. It follows that any vertex v is in H_{N_v} for some $N_v \in \mathbb{N}$. By definition of the H_i 's this implies that any path of length $\geq N_v$ starting at v must end in NW_G . □

Proposition 4.3 shows that NW_G works as a sort of black hole in G when G is cofinal.

Lemma 4.4. *Assume that G is cofinal. No vertex $v \in V \setminus NW_G$ is an infinite emitter.*

Proof. Let $v \in V$ be an infinite emitter. Set

$$A = \{w \in V : w = s(\mu) \text{ for some } \mu \in P_f(G) \text{ such that } r(\mu) = v\}.$$

Then $v \in A$, and since $V \setminus A$ is hereditary and saturated, it follows that $A = V$. In particular, $r(s^{-1}(v)) \subseteq A$, which implies that $v \in NW_G$. \square

In the following we will say that *all powers of $A(\beta)$ are finite* when $A(\beta)_{uv}^n < \infty$ for all $n \in \mathbb{N}$ and all $u, v \in V$.

Lemma 4.5. *Assume that G is cofinal. Let $\psi \in [0, \infty]^V$ be a $A(\beta)$ -harmonic vector. Assume that ψ is not identical zero. Then $\psi_v > 0$ for all $v \in V$ and all powers of $A(\beta)$ are finite.*

Proof. That ψ must be strictly positive follows from the observation that

$$\{v \in V : \psi_v = 0\}$$

is hereditary and saturated. It follows then from (2.6) that all powers of $A(\beta)$ are finite. \square

Lemma 4.6. *Assume that G is cofinal. Let $\beta \in \mathbb{R}$.*

- 1) *A sink $v \in V_\infty$ is β -summable.*
- 2) *An infinite emitter $v \in V_\infty$ is β -summable if and only if $\sum_{n=0}^\infty A(\beta)_{vv}^n < \infty$. This condition implies that all powers of $A(\beta)$ are finite.*

Proof. 1) Let $w \in V$. It follows from Corollary 4.2 that there is an $N_w \in \mathbb{N}$ such that $\sum_{n=0}^\infty A(\beta)_{vw}^n = \sum_{n=0}^{N_w} A(\beta)_{vw}^n$. This sum is finite since $V = V \setminus NW_G$ by 2) of Corollary 4.2 and $V \setminus NW_G$ contains no infinite emitter by Lemma 4.4.

2) The set

$$\left\{ u \in V : \sum_{n=0}^\infty A(\beta)_{uv}^n < \infty \right\} \quad (4.1)$$

is hereditary and saturated. Cofinality of G therefore implies the first statement. To prove the second statement, consider two vertexes $w, u \in V$ and a $k \in \mathbb{N}$. It follows from Lemma 4.4 that $v \in NW_G$. In particular, NW_G is not empty and hence 2) of Corollary 4.2 implies that there are no sinks. By Proposition 4.3 NW_G is a strongly connected subgraph of G whose vertexes constitute a hereditary subset of V , and it follows therefore from Lemma 4.1 that there is an $l \in \mathbb{N}$ such that $A(\beta)_{uv}^l \neq 0$. Since

$$A(\beta)_{wu}^k A(\beta)_{uv}^l \leq A(\beta)_{wv}^{k+l} < \infty,$$

it follows that $A(\beta)_{wu}^k < \infty$. \square

Corollary 4.7. *Assume that G is cofinal. There are no non-zero β -KMS measures unless all powers of $A(\beta)$ are finite.*

Proof. It follows from Lemma 4.5 and Theorem 3.8 that there are no non-zero harmonic β -KMS measures unless all powers of $A(\beta)$ are finite. It follows from 2) of Lemma 4.6 that there are no non-zero boundary β -KMS measures unless all powers of $A(\beta)$ are finite. \square

In view of Corollary 4.7 we shall in the following implicitly assume that all powers of $A(\beta)$ are finite.

4.1. No non-wandering vertexes. We split now the considerations into three cases, depending of the size of NW_G . We begin with the case where $NW_G = \emptyset$. Then G has no infinite emitters by Lemma 4.4 and at most one sink by 1) in Lemma 4.2. In particular, G is row-finite and except for the possible presence of a sink, the case is covered by Corollary 7.3 in [Th2].

Theorem 4.8. *Assume that G is cofinal and that NW_G is empty.*

- a) *Assume that G contains a sink. For every $\beta \in \mathbb{R}$ there is a β -KMS weight for α^F . It is unique up multiplication by scalars, and is given by the boundary β -KMS measure of the sink.*
- b) *Assume that there is no sink in G . For any $\beta \in \mathbb{R}$ there are β -KMS weights, and they are all harmonic and gauge invariant.*

Proof. $C^*(G)$ is simple in this case by Theorem 12 in [Sz] and it follows from Proposition 5.6 in [CT2] that all KMS weights are gauge invariant. a): It follows from Corollary 4.2 that there is only one sink and that $P(G) = \emptyset$. In particular, there are no non-zero harmonic β -KMS measures. There are no infinite emitters in G by Lemma 4.4 and the sink is therefore the only element in V_∞ . Since the sink is β -summable by 1) of Lemma 4.6, the statements follow from Theorems 2.2, 2.7 and 3.4. b): This case is covered by Corollary 7.3 in [Th2]. \square

In general, the β -KMS weights in case b) are not unique, not even up to multiplication by scalars. In fact, work of Kishimoto, [Ki], suggests that the KMS spectrum can be very rich, but we will not pursue this case further in the present work.

4.2. A reduction. We want to study the KMS weights of α^F when G is cofinal and NW_G not empty, and in this section we show that for this purpose we can assume that G is strongly connected.

Let G be a cofinal graph and assume that $NW_G \neq \emptyset$. Then NW_G is a strongly connected subgraph of G by Proposition 4.3. Note that

$$U = \{x \in \Omega_G : s(x) \in NW_G\}$$

is a closed and open subset of Ω_G which we can and will identify with Ω_{NW_G} . The reduction

$$\mathcal{G}|_U = \{\xi \in \mathcal{G} : s(\xi) \in U, r(\xi) \in U\}$$

is an étale locally compact second countable Hausdorff groupoid and the corresponding convolution C^* -algebra $C_r^*(\mathcal{G}|_U)$ is isomorphic to $C^*(NW_G)$. In this way we get an embedding

$$C^*(NW_G) \subseteq C^*(G)$$

of $C^*(NW_G)$ as a hereditary C^* -subalgebra of $C^*(G)$, cf. Proposition 1.9 in [Ph]. Note that α^F leaves $C^*(NW_G)$ globally invariant.

Proposition 4.9. *Assume that G is cofinal and that $NW_G \neq \emptyset$. For each $\beta \in \mathbb{R}$ the restriction $\psi \mapsto \psi|_{C^*(NW_G)}$ is a bijection from the β -KMS weights for α^F on $C^*(G)$ to those for the restriction of α^F to $C^*(NW_G)$.*

Proof. In view of Theorem 2.4 it suffices to take a vertex $v \in NW_G$ and show that P_v is full in both $C^*(G)$ and $C^*(NW_G)$. Let I be an ideal in $C^*(G)$ containing P_v . Then $\{w \in V : P_w \in I\}$ is not empty. If $e \in E$ and $P_{s(e)} \in I$ we find that $P_{r(e)} = S_e^* S_e \in I$, and if $v \in V \setminus V_\infty$ and $P_{r(e)} \in I$ for all $e \in s^{-1}(v)$ we find that $P_v = \sum_{e \in s^{-1}(v)} S_e S_e^* = \sum_{e \in s^{-1}(v)} S_e P_{r(e)} S_e^* \in I$. It follows then from the cofinality of G that $\{w \in V : P_w \in I\} = V$ and hence that $I = C^*(G)$. Thus P_v is full in $C^*(G)$. Since NW_G is strongly connected an even simpler argument shows P_v is also full in $C^*(NW_G)$. \square

In view of Proposition 4.9 and since we have handled the case when $NW_G = \emptyset$ in Section 4.1, we can restrict attention to $C^*(NW_G)$. Thus in the sequel we shall assume that G is strongly connected. But then $C^*(G)$ is simple by Theorem 12 in [Sz], except when G consists of a single loop and nothing more. In that case all KMS weights for α^F can be normalized to states and they can be found and described by use of results from [CT1]. There is therefore no loss in assuming that G is strongly connected and that $C^*(G)$ is simple. This will simplify many formulations in the following because all KMS weights for α^F are then gauge invariant by Proposition 5.6 in [CT2].

4.3. A Hopf dichotomy. In [CL] Carlsen and Larsen introduced an interesting division of the harmonic KMS states on a general graph C^* -algebra. It is this division we study in this section, but for KMS weights on strongly connected graphs. The goal is to show that harmonic β -KMS measures are always either dissipative or conservative, in a sense we now make precise. We assume throughout that G is strongly connected. Set

$$P(G)_{rec} = \bigcap_{e \in E} \{(x_i)_{i=1}^\infty \in P(G) : x_i = e \text{ for infinitely many } i\}, \quad (4.2)$$

and

$$P(G)_{wan} = \bigcap_{v \in V} \{(x_i)_{i=1}^\infty \in P(G) : \#\{i \in \mathbb{N} : s(x_i) = v\} < \infty\}. \quad (4.3)$$

A Borel measure m on $P(G)$ is *conservative* when $m(P(G) \setminus P(G)_{rec}) = 0$, and *dissipative* when $m(P(G) \setminus P(G)_{wan}) = 0$. These notions stem from the theory of dynamical systems where they describe important properties of the measure m with respect to the left shift on $P(G)$. The aim is here to show how these properties can be determined from the matrix $A(\beta)$.

We will say that $A(\beta)$ is *recurrent* when $\sum_{n=0}^\infty A(\beta)_{vv}^n = \infty$ for some $v \in V$. The inequality

$$A(\beta)_{vu}^l A(\beta)_{uw}^n A(\beta)_{wv}^k \leq A(\beta)_{vv}^{n+k+l}, \quad (4.4)$$

valid for all $n, k, l \in \mathbb{N}$ and all $u, v, w \in V$, shows that because G is strongly connected $A(\beta)$ is recurrent if and only if $\sum_{n=0}^\infty A(\beta)_{uw}^n = \infty$ for all $u, w \in V$. We say that $A(\beta)$ is *transient* when $\sum_{n=0}^\infty A(\beta)_{vv}^n < \infty$ for one $v \in V$, or equivalently that $\sum_{n=0}^\infty A(\beta)_{vv}^n < \infty$ for all $v, w \in V$.

Theorem 4.10. *Assume that G is strongly connected, and let m be a non-zero harmonic β -KMS measure on $P(G)$. Then m is conservative if and only if $A(\beta)$ is recurrent, and dissipative if and only if $A(\beta)$ is transient.*

Proof. Set $\psi_v = m(Z'(v))$ and note that ψ is $A(\beta)$ -harmonic by Lemma 2.6 and consider the corresponding matrix $B(\beta)$ from (3.6). By Lemma 4.5 $B(\beta)$ is a matrix over E in this case. It is easy to show (by induction) that

$$B(\beta)_{e,f}^n = \psi_{r(e)}^{-1} \psi_{r(f)} e^{-\beta F(f)} A(\beta)_{r(e),s(f)}^{n-1} \quad (4.5)$$

for all $n \geq 1$. With this as one key of translation, much of the proof can be completed by use of standard results from the theory of Markov chains as follows. We use Theorems 3.2 and 3.4 in [Wo] to the Markov chains on E arising from $B(\beta)$. For this note that when we consider an edge $e \in E$ as a state x for such a Markov chain, the measure \Pr_x in [Wo] is the measure \Pr_e used in the proof in Lemma 3.6, cf.(3.8), and the quantity $H(x, y)$ in [Wo] is

$$\Pr_x(\{(x_i)_{i=1}^\infty \in P(G) : x_i = y \text{ for infinitely many } i\}).$$

Equally important is it that the quantity $G(x, x)$ occurring in Theorem 3.4 of [Wo] in the present application is the sum $\sum_{n=0}^\infty B(\beta)_{x,x}^n$. Once this is realized the rest of the 'translation' reads as follows.

Assume that m is conservative. Fix an edge $e \in E$. Since m is conservative it is supported on the set (4.2) and since $m = m_\psi$ by Theorem 3.8, it follows that so is the measure \Pr_e , and the edge e is therefore recurrent as a state for the Markov chain defined from $B(\beta)$, cf. Definition 3.1 in [Wo]. It follows first from Theorem 3.4 (a) in [Wo] that $\sum_{n=0}^\infty B(\beta)_{e,e}^n = \infty$ and then from (4.5) (and Lemma 4.5) that $A(\beta)$ is recurrent. Conversely, if $A(\beta)$ is recurrent it follows from (4.5) and Theorem 3.4 (a) in [Wo] that e is a recurrent state for the Markov chain. Since G is strongly connected it follows then from Theorem 3.4 (b) in [Wo] that

$$\Pr_e(\{(x_i)_{i=1}^\infty \in Z'(e) : x_i = f \text{ for infinitely many } i\}) = 1$$

for all $f \in E$. But E is countable and \Pr_e is a probability measure on $Z'(e)$ and it follows therefore that

$$\Pr_e\left(\bigcap_{f \in E} \{(x_i)_{i=1}^\infty \in Z'(e) : x_i = f \text{ for infinitely many } i\}\right) = 1. \quad (4.6)$$

Now recall from the proof of Lemma 3.7 that

$$m(B) = m_\psi(B) = \sum_{v \in V} \sum_{e \in s^{-1}(v)} e^{-\beta F(e)} \psi_{r(e)} \Pr_e(B \cap Z'(e)) \quad (4.7)$$

for all Borel sets B . By combining (4.6) and (4.7) it follows that $m(P(G) \setminus P(G)_{rec}) = 0$, i.e. m is conservative.

Assume then that m is dissipative. Note that

$$P(G)_{wan} \subseteq \bigcap_{e \in E} \{(x_i)_{i=1}^\infty \in P(G) : \#\{i \in \mathbb{N} : x_i = e\} < \infty\}.$$

By combining this with the relation (4.7) it follows from Theorem 3.2 (b) in [Wo] that every edge $e \in E$ is a transient state for the Markov chain defined from $B(\beta)$. By Theorem 3.4 (a) in [Wo] this means that $\sum_{n=0}^\infty B(\beta)_{e,e}^n < \infty$ for all $e \in E$, and then $A(\beta)$ is transient by (4.5).

The converse implication, that transience of $A(\beta)$ implies dissipativity of m , is stronger than what can be obtained directly from [Wo] by the method used so far

and we give instead a direct proof. Consider two vertexes w, v and the set

$$M(w, v) = \{(x_i)_{i=1}^\infty \in Z'(w) : s(x_j) = v \text{ for infinitely many } j\}.$$

Let $N \in \mathbb{N}$ and for $j \geq N$, let

$$M^j = \{(x_i)_{i=1}^\infty \in Z'(w) : s(x_j) = v\}.$$

Then $M(w, v) \subseteq \bigcup_{j \geq N} M^j$ and hence

$$m(M(w, v)) \leq \sum_{j=N}^\infty m(M^j).$$

From Lemma 2.8 we get the identity $m(M^j) = A(\beta)_{wv}^{j-1} m(Z'(v))$ and therefore the estimate

$$m(M(w, v)) \leq m(Z'(v)) \sum_{j=N}^\infty A(\beta)_{wv}^{j-1}.$$

Since $\lim_{N \rightarrow \infty} \sum_{j=N}^\infty A(\beta)_{wv}^{j-1} = 0$ because $A(\beta)$ is transient, we conclude that $m(M(w, v)) = 0$. This completes the proof because

$$P(G) \setminus P(G)_{\text{wan}} = \bigcup_{w, v \in V} M(w, v).$$

□

Note that Theorem 4.10 describes a dichotomy: For any β either all non-zero harmonic β -KMS measures are dissipative or they are all conservative.

4.4. Criteria for recurrence and transience. In Proposition 4.12 below we describe a recurrence/transience criterion which will be very useful in the following. It is essentially equivalent to part (i) of Theorem C in [V], but while Vere-Jones omits the proof by referring to standard renewal arguments, we give a direct proof, parts of which will be used later.

Let $v \in V$. A *loop* based at v in G is a finite path $\mu = e_1 e_2 e_3 \cdots e_n$ of length $n \geq 1$ such that $s(\mu) = r(\mu) = v$. It is a *simple loop* when $r(e_i) \neq r(\mu)$, $i = 1, 2, \dots, n-1$. The set of loops (resp. simple loops) μ of length n based at v will be denoted by $L_v(n)$ (resp. $l_v(n)$). We set

$$l_{vv}^n(\beta) = \sum_{\mu \in l_v(n)} e^{-\beta F(\mu)}$$

when $l_v(n) \neq \emptyset$ and $l_{vv}^n(\beta) = 0$ when $l_v(n) = \emptyset$.

Lemma 4.11. *For $t > 0$,*

$$\sum_{k=0}^\infty \left(\sum_{j=1}^\infty l_{vv}^j(\beta) t^j \right)^k = \sum_{n=0}^\infty A(\beta)_{vv}^n t^n.$$

Proof. We claim that

$$\sum_{n=1}^N A(\beta)_{vv}^n t^n \leq \sum_{k=1}^N \left(\sum_{j=1}^N l_{vv}^j(\beta) t^j \right)^k \leq \sum_{n=1}^{N^2} A(\beta)_{vv}^n t^n \quad (4.8)$$

for all $N \in \mathbb{N}$. The proof is a matter of book-keeping. For each $(d_1, d_2, \dots, d_j) \in \{1, 2, \dots, N\}^j$, set

$$W(d_1, d_2, \dots, d_j) = l_{vv}^{d_1}(\beta) t^{d_1} l_{vv}^{d_2}(\beta) t^{d_2} \dots l_{vv}^{d_j}(\beta) t^{d_j},$$

and

$$\{1, 2, \dots, N\}_W^j = \left\{ \xi \in \{1, 2, \dots, N\}^j : W(\xi) \neq 0 \right\}.$$

Set

$$A_N = \sqcup_{k=1}^N \{1, 2, \dots, N\}_W^k,$$

and note that

$$\sum_{k=1}^N \left(\sum_{j=1}^N l_{vv}^j(\beta) t^j \right)^k = \sum_{\xi \in A_N} W(\xi). \quad (4.9)$$

An element $\mu \in \sqcup_{n=1}^N L_v(n)$ admits a unique decomposition $\mu = \mu_1 \mu_2 \dots \mu_j$ such that each μ_i is a simple loop based at v , and in this way μ determines an element

$$p(\mu) = (|\mu_1|, |\mu_2|, \dots, |\mu_j|) \in A_N.$$

Since

$$p(L_v(n)) = \sqcup_{k=1}^N \{(d_1, d_2, \dots, d_k) \in \{1, 2, \dots, N\}_W^k : d_1 + \dots + d_k = n\}$$

and

$$\sum_{\mu \in p^{-1}(\xi)} e^{-\beta F(\mu)} t^{|\mu|} = W(\xi),$$

we find that

$$A(\beta)_{vv}^n t^n = \sum_{\mu \in L_v(n)} e^{-\beta F(\mu)} t^{|\mu|} = \sum_{\xi} W(\xi),$$

where the last sum is over

$$\xi \in \sqcup_{k=1}^N \{(d_1, d_2, \dots, d_k) \in \{1, 2, \dots, N\}_W^k : d_1 + \dots + d_k = n\}.$$

Consequently

$$\sum_{n=1}^N A(\beta)_{vv}^n t^n = \sum_{\xi \in B_N} W(\xi), \quad (4.10)$$

where

$$B_N = \sqcup_{k=1}^N \{(d_1, d_2, \dots, d_k) \in \{1, 2, \dots, N\}_W^k : d_1 + \dots + d_k \leq N\}.$$

Then (4.8) follows by combining (4.9) and (4.10) with the observation that $B_N \subseteq A_N \subseteq B_{N^2}$. Define functions $f_N : \mathbb{N} \rightarrow [0, \infty)$ such that

$$f_N(k) = \begin{cases} \left(\sum_{j=1}^N l_{vv}^j(\beta) \right)^k, & k \in \{1, 2, \dots, N\}, \\ 0, & k > N. \end{cases}$$

Since $f_N \leq f_{N+1}$, an application of Lebesgues monotone convergence theorem to this sequence of functions shows that

$$\lim_{N \rightarrow \infty} \sum_{k=1}^N \left(\sum_{j=1}^N l_{vv}^j(\beta) \right)^k = \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} l_{vv}^j(\beta) \right)^k.$$

Combined with the inequalities from (4.8) this leads to the stated identity. \square

Proposition 4.12. *Assume that G is strongly connected and let $\beta \in \mathbb{R}$.*

- $A(\beta)$ is recurrent if and only if $\sum_{k=1}^{\infty} l_{vv}^k(\beta) \geq 1$ for some (and hence every) vertex $v \in V$.
- $A(\beta)$ is transient if and only if $\sum_{k=1}^{\infty} l_{vv}^k(\beta) < 1$ for some (and hence every) vertex $v \in V$.

Proof. Take $t = 1$ in Lemma 4.11. \square

4.5. The recurrent case. We still assume that G is strongly connected. In order to describe for which β there are β -KMS weights we introduce the number (in $[-\infty, \infty]$),

$$\mathbb{P}(-\beta F) = \log \left(\limsup_n (A(\beta)_{vv}^n)^{\frac{1}{n}} \right), \quad (4.11)$$

where v is any vertex. Since G is strongly connected it follows from inequalities like (4.4) that this quantity does not depend on v and is equal to

$$\log \left(\limsup_n (A(\beta)_{uw}^n)^{\frac{1}{n}} \right)$$

for any pair of vertexes u, w . The quantity $\mathbb{P}(-\beta F)$ appears in the theory of dynamical systems. In fact it is a version of what O. Sarig calls the *Gurevich pressure* in [S] and the function $\beta \mapsto \mathbb{P}(-\beta F)$ is sometimes called the pressure function corresponding to a potential defined from F . When G is finite, and also in certain cases with G infinite, $\mathbb{P}(-\beta F)$ is the logarithm of the spectral radius of a bounded operator defined from $A(\beta)$.

Lemma 4.13. *Assume that G is strongly connected and that $C^*(G)$ is simple. Let $\beta \in \mathbb{R}$. There are no β -KMS weights unless $\mathbb{P}(-\beta F) \leq 0$ and no boundary β -KMS weights unless $A(\beta)$ is transient.*

Proof. If $A(\beta)$ is recurrent it follows that $\sum_{n=0}^{\infty} A(\beta)_{vv}^n = \infty$ for all $v, w \in V$ and hence no infinite emitter can be β -summable. By Theorem 3.4 there are then no boundary β -KMS weights. Since $A(\beta)$ is recurrent if $\mathbb{P}(-\beta F) > 0$ this also implies that there are no boundary β -KMS weights unless $\mathbb{P}(-\beta F) \leq 0$. If there is a harmonic β -KMS weight it follows from Theorem 3.8 that there is a non-zero $A(\beta)$ -harmonic vector ψ . But then $A(\beta)_{vv}^n \psi_v \leq \sum_w A(\beta)_{vw}^n \psi_w \leq \psi_v$ for all $n \in \mathbb{N}$, $v \in V$. Since $\psi_v > 0$ by Lemma 4.5 it follows that

$$\limsup_n (A(\beta)_{vv}^n)^{\frac{1}{n}} = \limsup_n (A(\beta)_{vv}^n \psi_v)^{\frac{1}{n}} \leq \limsup_n \psi_v^{\frac{1}{n}} = 1,$$

implying that $\mathbb{P}(-\beta F) \leq 0$. It follows now from Theorems 2.2 and 2.7 that there are no gauge invariant β -KMS weights when $\mathbb{P}(-\beta F) > 0$. By Proposition 5.6 in [CT2] there are then no β -KMS weights at all. \square

Theorem 4.14. *Assume that G is strongly connected and that $C^*(G)$ is simple. Let $\beta \in \mathbb{R}$ and assume that $A(\beta)$ is recurrent. There exists a β -KMS weight for α^F if and only if $\mathbb{P}(-\beta F) = 0$. When this is the case the β -KMS weight is unique up to multiplication by scalars, it is harmonic and the corresponding β -KMS measure on $P(G)$ is conservative.*

Proof. Assume that there is a β -KMS weight for α^F . Since $A(\beta)$ is transient when $\mathbb{P}(-\beta F) < 0$ it follows from Lemma 4.13 that $\mathbb{P}(-\beta F) = 0$. Assume then that $\mathbb{P}(-\beta F) = 0$. By Lemma 4.13 there are no boundary β -KMS weights, but the work of Vere-Jones shows that there is a non-zero $A(\beta)$ -harmonic vector, and that

it is unique up to multiplication by scalars. This follows from Corollary 2 on page 371 in [V], and from Corollary 3.9 we conclude that there is a gauge invariant β -KMS weight for α^F which is unique up to multiplication by scalars. It follows from Theorem 4.10 that the corresponding β -KMS measure is conservative. Finally, by Proposition 5.6 in [CT2] all β -KMS weights are gauge invariant. \square

The following lemma gives a useful criterion for when the assumptions in Theorem 4.14 are satisfied.

Lemma 4.15. *Assume that G is strongly connected and let $\beta \in \mathbb{R}$. The following are equivalent:*

- 1) $\sum_{k=1}^{\infty} l_{vv}^k(\beta) = 1$.
- 2) $A(\beta)$ is recurrent and $\mathbb{P}(-\beta F) = 0$.

Proof. 1) \Rightarrow 2): The recurrence of $A(\beta)$ is a direct consequence of Proposition 4.12 and 1). To show that $\mathbb{P}(-\beta F) = 0$, note that $e^{-\mathbb{P}(-\beta F)}$ is the radius of convergence of the power series $\sum_{n=0}^{\infty} A(\beta)_{vv}^n t^n$. By Lemma 4.11 this is the same as

$$\sup \left\{ t \geq 0 : \sum_{k=1}^{\infty} l_{vv}^k(\beta) t^k < 1 \right\}.$$

It follows from 1) that this number is 1.

2) \Rightarrow 1): Since $A(\beta)$ is recurrent it follows from Proposition 4.12 that $\sum_{k=1}^{\infty} l_{vv}^k(\beta) \geq 1$. Since we also assume that $\mathbb{P}(-\beta F) = 0$, which means that the radius of the convergence of the power series $\sum_{n=0}^{\infty} A(\beta)_{vv}^n t^n$ is 1, it follows first that $\sum_{n=0}^{\infty} A(\beta)_{vv}^n t^n < \infty$ and then by Lemma 4.11 that $\sum_{k=1}^{\infty} l_{vv}^k(\beta) t^k < 1$ when $t < 1$. From Lebesgue's theorem on monotone convergence we deduce from this that

$$\sum_{k=1}^{\infty} l_{vv}^k(\beta) = \lim_{t \uparrow 1} \sum_{k=1}^{\infty} l_{vv}^k(\beta) t^k \leq 1.$$

\square

In view of Theorem 4.14 we shall in the following focus on the case where $A(\beta)$ is transient. Concerning the boundary KMS-weights the situation is rather simple:

Proposition 4.16. *Assume that G is strongly connected and that $C^*(G)$ is simple. If $A(\beta)$ is transient there is a bijective correspondence between the infinite emitters in G and the extremal rays of boundary β -KMS weights.*

Proof. Since $A(\beta)$ is transient it follows from 2) of Lemma 4.6 that all infinite emitters are β -summable and then from Theorem 3.4 that the infinite emitters correspond bijectively to the extremal rays of boundary β -KMS weights. \square

4.6. Finitely many non-wandering vertexes. In this section we now consider the case where G is strongly connected and only has finitely many vertexes. It follows from 2) of Corollary 4.2 that there are no sinks in G , but there may be infinite emitters in V .

Theorem 4.17. *Assume that G strongly connected with finitely many vertexes and that $C^*(G)$ is simple. Let $\beta \in \mathbb{R}$. Assume that $A(\beta)$ is transient. The extremal rays of β -KMS weights for α^F are in bijective correspondence with the infinite emitters in G , and there are no harmonic β -KMS weights.*

Proof. As usual it follows from Proposition 5.6 in [CT2] that all KMS-weights are gauge invariant, and in view of Proposition 4.16 it suffices therefore to show that there are no non-zero $A(\beta)$ -harmonic vectors. So assume ψ is such a vector, and let v be a vertex in G . Then $\psi_v > 0$ by Lemma 4.5 and hence

$$\sum_{w \in G} \sum_{n=0}^{\infty} A(\beta)_{vw}^n \psi_w = \sum_{n=0}^{\infty} \sum_{w \in G} A(\beta)_{vw}^n \psi_w = \sum_{n=0}^{\infty} \psi_v = \infty.$$

It follows that $\sum_{n=0}^{\infty} A(\beta)_{vw}^n = \infty$ for at least one, and hence for all $w \in V$ since G is strongly connected. In particular, $\sum_{n=0}^{\infty} A(\beta)_{vv}^n = \infty$ which contradicts the assumed transience of $A(\beta)$. \square

In the setting of Theorem 4.17 the algebra $C^*(G)$ is unital and all KMS-weights can be normalized to states.

Theorem 4.17 applies of course also when G is a finite graph which is strongly connected such that $C^*(G)$ is simple. In combination with Theorem 4.14 the conclusion is that there is at most one KMS state in this case. For the gauge action, where $F = 1$, this was first proved by Enomoto, Fujii and Watatani, [EFW], and when F is strictly positive (or strictly negative) it follows from the work of Exel and Lacas, [EL]. For general F it follows from [CT1] where the necessary and sufficient conditions for the existence of the KMS state is also described. As the next example shows the presence of infinite emitters allows for a more complicated KMS spectrum.

Example 4.18. Let G be the directed graph with two vertexes v and w and edges e_i, f_i and a, b , such that $r(e_i) = s(e_i) = v$, $r(f_i) = s(f_i) = w$ for all $i \in \mathbb{N}$, while $s(a) = v, r(a) = w$ and $s(b) = w$ and $r(b) = v$. Define $F : E \rightarrow \mathbb{R}$ such that $F(e_i) = F(f_i) = \log(i + 1)$ for all i and $F(a) = F(b) = 1$. For $\beta > 1$, set

$$s(\beta) = \sum_{i=2}^{\infty} i^{-\beta}.$$

Then

$$l_{vv}^i(\beta) = \begin{cases} s(\beta), & i = 1, \\ e^{-2\beta}, & i = 2, \\ s(\beta)^{i-2} e^{-2\beta}, & i \geq 3, \end{cases}$$

and hence

$$\sum_{k=1}^{\infty} l_{vv}^k(\beta) = s(\beta) + e^{-2\beta} + \frac{s(\beta)e^{-2\beta}}{1 - s(\beta)}$$

when $s(\beta) < 1$. It follows that there is a real number $\beta_0 > 0$ such that $\sum_{k=1}^{\infty} l_{vv}^k(\beta) < 1$ when $\beta > \beta_0$, $\sum_{k=1}^{\infty} l_{vv}^k(\beta_0) = 1$ and $\sum_{k=1}^{\infty} l_{vv}^k(\beta) > 1$ when $\beta < \beta_0$. Hence $A(\beta)$ is transient if and only if $\beta > \beta_0$ by Proposition 4.12. Theorem 4.17 now tells us that there are two extremal β -KMS states for α^F when $\beta > \beta_0$; one for each of two vertexes since they are both infinite emitters. Theorem 4.14 implies that there are no β -KMS states when $\beta < \beta_0$, and exactly one when $\beta = \beta_0$.

We have now obtained a complete description and numbering of the β -KMS weights when G is strongly connected, $C^*(G)$ is simple and either $A(\beta)$ is recurrent or G only has finitely many vertexes. What remains is therefore the cases where $A(\beta)$ is transient and G has infinitely many vertexes. The only general result we have to offer in that case is the following.

Proposition 4.19. *Assume that G is strongly connected with infinitely many vertices and that $C^*(G)$ is simple. For $\beta \in \mathbb{R}$ there exist β -KMS weights for α^F if and only if $\mathbb{P}(-\beta F) \leq 0$.*

Proof. Thanks to Lemma 4.13 there is only one implication to consider. So assume that $\mathbb{P}(-\beta F) \leq 0$. If $A(\beta)$ is recurrent we have that $\mathbb{P}(-\beta F) = 0$ since $\sum_{n=0}^{\infty} A(\beta)_{vv}^n < \infty$ if $\mathbb{P}(-\beta F) < 0$. It follows therefore from Theorem 4.14 that there is a β -KMS weight in this case. Assume then that $A(\beta)$ is transient. If there are infinite emitters in G the existence of a β -KMS weight follows from Proposition 4.16. Finally, when $A(\beta)$ is transient and G does not have infinite emitters, it is a result of Pruitt, found as the Corollary at the bottom of page 1799 in [Pr], that there is a non-zero $A(\beta)$ -harmonic vector because $\mathbb{P}(-\beta F) \leq 0$. By Corollary 3.9 there is therefore a β -KMS weight also in this case. \square

We know now, at least in principle, when there are any β -KMS weights to be found. In the following we shall identify a class of strongly connected graphs where the structure of KMS weights is both transparent and very rich.

5. KMS WEIGHTS FROM EXITS IN G

In this section we assume that G is strongly connected and that $A(\beta)$ is transient. Set

$$P(V) = \{(v_i)_{i=1}^{\infty} \in V^{\mathbb{N}} : v_{i+1} \in r(s^{-1}(v_i)) \ \forall i\}.$$

An element $t = (t_i)_{i=1}^{\infty} \in P(V)$ will be called an *exit path* when $\lim_{i \rightarrow \infty} t_i = \infty$, in the sense that for every finite subset $M \subseteq V$ there is an $N \in \mathbb{N}$ such that $t_i \notin M$ when $i \geq N$. Let $\beta \in \mathbb{R}$ and consider an exit path $t = (t_i)_{i=1}^{\infty} \in P(V)$. Set

$$t^{\beta}(i) = A(\beta)_{t_1 t_2} A(\beta)_{t_2 t_3} \cdots A(\beta)_{t_{i-1} t_i}.$$

Then $\sum_{n=0}^{\infty} A(\beta)_{vt_i}^n < \infty$ for all i since $A(\beta)$ is transient and

$$t^{\beta}(i)^{-1} \sum_{n=0}^{\infty} A(\beta)_{vt_i}^n = t^{\beta}(i+1)^{-1} \sum_{n=0}^{\infty} A(\beta)_{vt_i}^n A(\beta)_{t_i t_{i+1}} \leq t^{\beta}(i+1)^{-1} \sum_{n=0}^{\infty} A(\beta)_{vt_{i+1}}^n$$

for all i and all $v \in V$. Hence the limit

$$\lim_{i \rightarrow \infty} t^{\beta}(i)^{-1} \sum_{n=0}^{\infty} A(\beta)_{vt_i}^n \tag{5.1}$$

exists, although it may of course be $+\infty$. For all $w \in V$ and $k, i \in \mathbb{N}$ we have the estimate

$$A(\beta)_{wv}^k t^{\beta}(i)^{-1} \sum_{n=0}^{\infty} A(\beta)_{vt_i}^n \leq t^{\beta}(i)^{-1} \sum_{n=0}^{\infty} A(\beta)_{wt_i}^{n+k} \leq t^{\beta}(i)^{-1} \sum_{n=0}^{\infty} A(\beta)_{wt_i}^n.$$

Because G is strongly connected it follows from this that the limit (5.1) is finite for all $v \in V$ if it is finite for one. When this holds we say that t is β -summable.

Lemma 5.1. *Let $\beta \in \mathbb{R}$ and assume that t is a β -summable exit path in G . It follows that the vector $\psi \in]0, \infty[^V$ defined such that*

$$\psi_v = \lim_{i \rightarrow \infty} t^{\beta}(i)^{-1} \sum_{n=0}^{\infty} A(\beta)_{vt_i}^n,$$

is $A(\beta)$ -harmonic.

Proof. Note that $\psi_{t_1} \geq 1$ since

$$\sum_{n=0}^{\infty} A(\beta)_{t_1 t_i}^n \geq A(\beta)_{t_1 t_i}^{i-1} \geq t^\beta(i)$$

when $i \geq 2$. Since $t^\beta(i)^{-1} \sum_{n=0}^{\infty} A(\beta)_{vt_i}^n$ never decreases when i increases, we find that

$$\begin{aligned} \sum_{w \in V} A(\beta)_{vw} \psi_w &= \lim_{i \rightarrow \infty} \sum_{w \in V} A(\beta)_{vw} t^\beta(i)^{-1} \sum_{n=0}^{\infty} A(\beta)_{wt_i}^n \\ &= \lim_{i \rightarrow \infty} t^\beta(i)^{-1} \sum_{n=0}^{\infty} \sum_{w \in V} A(\beta)_{vw} A(\beta)_{wt_i}^n = \lim_{i \rightarrow \infty} t^\beta(i)^{-1} \sum_{n=0}^{\infty} A(\beta)_{vt_i}^{n+1}. \end{aligned}$$

Since t is an exit path, it follows that $A(\beta)_{vt_i}^0 = 0$ and $\sum_{n=0}^{\infty} A(\beta)_{vt_i}^{n+1} = \sum_{n=0}^{\infty} A(\beta)_{vt_i}^n$ for all large i . Hence

$$\sum_{w \in V} A(\beta)_{vw} \psi_w = \lim_{i \rightarrow \infty} t^\beta(i)^{-1} \sum_{n=0}^{\infty} A(\beta)_{vt_i}^{n+1} = \psi_v.$$

□

It follows from Lemma 5.1 and Theorem 3.8 that a β -summable exit path t gives rise to a unique harmonic β -KMS measure m_t determined by the requirement that

$$m_t(Z'(v)) = \lim_{i \rightarrow \infty} t^\beta(i)^{-1} \sum_{n=0}^{\infty} A(\beta)_{vt_i}^n$$

for all $v \in V$. We call this an *exit measure*. To learn more about this measure we shall use the shift map σ which acts on both $P(V)$ and $P(G)$ in the natural way; in both cases given by the formula

$$\sigma((x_i)_{i=1}^{\infty}) = (x_{i+1})_{i=1}^{\infty}.$$

There is a map $\pi : P(G) \rightarrow P(V)$ defined by

$$\pi((e_i)_{i=1}^{\infty}) = (s(e_i))_{i=1}^{\infty},$$

which intertwines the two shift maps; $\sigma \circ \pi = \pi \circ \sigma$.

Lemma 5.2. *Let $t = (t_i)_{i=1}^{\infty} \in P(V)$. Then $\pi^{-1}(t)$ is a Borel subset of $P(G)$ and for any harmonic β -KMS measure m on $P(G)$, any vertex $v \in V$ and any $n \in \mathbb{N} \cup \{0\}$,*

$$m(Z'(v) \cap \sigma^{-n}(\pi^{-1}(t))) = \lim_{k \rightarrow \infty} A(\beta)_{vt_1}^n A(\beta)_{t_1 t_2} A(\beta)_{t_2 t_3} \cdots A(\beta)_{t_k t_{k+1}} m(Z'(t_{k+1})).$$

Proof. For each k let M_k be the set of paths $\mu = e_1 e_2 \cdots e_k$ in G such that $s(e_i) = t_i, i = 1, 2, \dots, k$, and $r(e_k) = t_{k+1}$. Then

$$\pi^{-1}(t) = \bigcap_{k \geq 1} \bigcup_{\mu \in M_k} Z'(\mu),$$

proving that $\pi^{-1}(t)$ is Borel. Let M_0 be the set of paths ν of length n from v to t_1 . Then

$$Z'(v) \cap \sigma^{-n}(\pi^{-1}(t)) = \bigcap_{k \geq 1} \left(\bigcup_{\nu \in M_0, \mu \in M_k} Z'(\nu \mu) \right).$$

Furthermore, since

$$\bigcup_{\mu \in M_{k+1}} Z'(\nu\mu) \subseteq \bigcup_{\mu \in M_k} Z'(\nu\mu)$$

for all k and all $\nu \in M_0$, it follows that

$$m(Z'(v) \cap \sigma^{-n}(\pi^{-1}(t))) = \lim_{k \rightarrow \infty} m\left(\bigcup_{\nu \in M_0, \mu \in M_k} Z'(\nu\mu)\right).$$

This completes the proof because it follows from (2.7) that

$$m\left(\bigcup_{\nu \in M_0, \mu \in M_k} Z'(\nu\mu)\right) = A(\beta)_{vt_1}^n A(\beta)_{t_1 t_2} A(\beta)_{t_2 t_3} \cdots A(\beta)_{t_k t_{k+1}} m(Z'(t_{k+1})).$$

□

Set

$$\mathcal{G}\pi^{-1}(t) := \bigcup_{n, m \in \mathbb{N}} \sigma^{-n}(\pi^{-1}(\sigma^m(t))).$$

Lemma 5.3. *Let $t \in P(V)$ be an exit path, and let m be a harmonic β -KMS measure on $P(G)$. Then*

$$m(Z'(v) \cap \mathcal{G}\pi^{-1}(t)) = m(\pi^{-1}(t)) \lim_{i \rightarrow \infty} t^\beta(i)^{-1} \sum_{n=0}^{\infty} A(\beta)_{vt_i}^n \quad (5.2)$$

for all $v \in V$.

Proof. Note that $\sigma^{-n}(\sigma^i(\pi^{-1}(t))) \subseteq \sigma^{-(n+1)}(\sigma^{i+1}(\pi^{-1}(t)))$ so that

$$\bigcup_{n=0}^{\infty} \sigma^{-n}(\sigma^i(\pi^{-1}(t))) \subseteq \bigcup_{n=0}^{\infty} \sigma^{-n}(\sigma^{i+1}(\pi^{-1}(t))).$$

Since

$$\mathcal{G}\pi^{-1}(t) = \bigcup_{i=1}^{\infty} \bigcup_{n=0}^{\infty} \sigma^{-n}(\sigma^i(\pi^{-1}(t))),$$

it follows that

$$m(Z'(v) \cap \mathcal{G}\pi^{-1}(t)) = \lim_{i \rightarrow \infty} m\left(Z'(v) \cap \bigcup_{n=0}^{\infty} \sigma^{-n}(\sigma^i(\pi^{-1}(t)))\right). \quad (5.3)$$

Because t is not pre-periodic under the shift, $\sigma^{-n}(\sigma^i(\pi^{-1}(t))) \cap \sigma^{-n'}(\sigma^i(\pi^{-1}(t))) = \emptyset$ when $n \neq n'$, and hence

$$m\left(Z'(v) \cap \bigcup_{n=0}^{\infty} \sigma^{-n}(\sigma^i(\pi^{-1}(t)))\right) = \sum_{n=0}^{\infty} m(Z'(v) \cap \sigma^{-n}(\sigma^i(\pi^{-1}(t)))). \quad (5.4)$$

Note that $\sigma^i(\pi^{-1}(t)) = \pi^{-1}(\sigma^i(t))$ and that $\sigma^i(t)$ is an exit path because t is. We can therefore use Lemma 5.2 to conclude that

$$m(Z'(v) \cap \sigma^{-n}(\pi^{-1}(\sigma^k(t)))) = A(\beta)_{vt_{k+1}}^n t^\beta(k+1)^{-1} m(\pi^{-1}(t)).$$

Inserted into (5.4) and (5.3) this yields (5.2). □

Lemma 5.4. *Let t be a β -summable exit path in $P(V)$ and m_t be the corresponding exit measure. Then $m_t(\pi^{-1}(t)) = 1$ and m_t is supported on $\mathcal{G}\pi^{-1}(t)$; that is, $m_t(P(G) \setminus \mathcal{G}\pi^{-1}(t)) = 0$.*

Proof. Set

$$U_n = \{(x_i)_{i=1}^\infty \in P(G) : s(x_i) = t_i, i \leq n\}.$$

Then $U_1 \supseteq U_2 \supseteq U_3 \supseteq \dots$ is a decreasing sequence of open sets in $P(G)$ such that

$$\bigcap_j U_j = \pi^{-1}(t).$$

It follows from Lemma 2.8 that

$$m_t(U_j) = t^\beta(j) \lim_{i \rightarrow \infty} t^\beta(i)^{-1} \sum_{n=0}^{\infty} A(\beta)_{t_j t_i}^n.$$

Since, for $i > j$,

$$\sum_{n=0}^{\infty} A(\beta)_{t_j t_i}^n \geq A(\beta)_{t_j t_{j+1}} A(\beta)_{t_{j+1} t_{j+2}} \cdots A(\beta)_{t_{i-1} t_i} = t^\beta(i) t^\beta(j)^{-1}$$

it follows that $m_t(U_j) \geq 1$ for all j . Combined with the observation that $m_t(U_j) \leq m_t(Z'(t_1)) < \infty$ for all j , we find that $1 \leq m_t(\pi^{-1}(t)) < \infty$. Define a Borel measure m on $P(G)$ such that

$$m(B) = m_t(\pi^{-1}(t))^{-1} m_t(B \cap \mathcal{G}\pi^{-1}(t)).$$

Note that $m(\pi^{-1}(t)) = 1$ and that m is supported on $\mathcal{G}\pi^{-1}(t)$. Since $\mathcal{G}\pi^{-1}(t)$ is \mathcal{G} -invariant it follows from Lemma 3.1 that m is a harmonic β -KMS measure. It follows therefore from Lemma 5.3 that

$$m(Z'(v)) = \lim_{i \rightarrow \infty} t^\beta(i)^{-1} \sum_{n=0}^{\infty} A(\beta)_{v t_i}^n = m_t(Z'(v))$$

for all $v \in V$. Hence $m = m_t$ by Theorem 3.8. \square

Corollary 5.5. *Let t be a β -summable exit path in $P(V)$, and let m_t be the corresponding exit measure. Then m_t is extremal among β -KMS measures. Specifically, when μ is a β -KMS measure such that $\mu \leq m_t$, then $\mu = s m_t$ where $s = \mu(\pi^{-1}(t))$.*

Proof. Assume μ is a β -KMS measure and that $\mu \leq m_t$. Since m_t is supported on $P(G)$ so is μ , and μ is therefore a harmonic β -KMS measure. Furthermore, μ is supported on $\mathcal{G}\pi^{-1}(t)$ since m_t is and it follows then from Lemma 5.3 that $\mu(Z'(v)) = \mu(\pi^{-1}(t)) m_t(Z'(v))$. This holds for all $v \in V$ and it follows from Theorem 3.8 that $\mu = \mu(\pi^{-1}(t)) m_t$. \square

Two elements $t = (t_i)_{i=1}^\infty, t' = (t'_i)_{i=1}^\infty$ in $P(V)$ are *tail equivalent* when there is a $k \in \mathbb{Z}$ such that $t_{i+k} = t'_i$ for all large i , and *tail inequivalent* otherwise. A tail equivalence class of exit paths will be called an *exit*. For $\beta \in \mathbb{R}$ we say that an exit is β -summable when one of its exit paths is β -summable (and then they all are).

Proposition 5.6. *Assume that G is strongly connected, that $C^*(G)$ is simple and that $A(\beta)$ is transient. Let m be a harmonic β -KMS measure on $P(G)$. There is a decomposition $m = m_1 + m_2$ where $m_i, i = 1, 2$, are harmonic β -KMS measures such that*

- i) there are a $N \in \mathbb{N} \cup \{0, \infty\}$, tail inequivalent β -summable exit paths t^1, t^2, \dots, t^N and positive real numbers $\lambda_1, \lambda_2, \dots, \lambda_N$ such that

$$m_1 = \sum_{i=1}^N \lambda_i m_{t^i},$$

and

- ii) $m_2 \circ \pi^{-1}(t) = 0$ for all $t \in P(V)$.

Proof. Set $M = \{t \in P(V) : m(\pi^{-1}(t)) > 0\}$. Since m is dissipative by Theorem 4.10 all elements of M are exit paths. Since $m(Z(v)) < \infty$ it follows that $M \cap \{t \in P(V) : t_1 = v\}$ is countable for all v , and hence that M is countable. Let $M = M_1 \sqcup M_2 \sqcup \dots$ be the partition of M into tail equivalence classes and take an element $t^i = (t_j^i)_{j=1}^\infty \in M_i$ for each i . It follows from Lemma 5.3 that

$$\lim_{j \rightarrow \infty} (t^i)^\beta(j)^{-1} \sum_{n=0}^{\infty} A(\beta)_{t_1^i t_j^i}^n \leq \frac{m(Z'(t_1^i))}{m(\pi^{-1}(t^i))},$$

showing that the t^i 's are all β -summable. Since $B \mapsto m(B \cap \mathcal{G}\pi^{-1}(t^i))$ is a harmonic β -KMS measure by Lemma 3.1 it follows from a combination of Lemma 5.3 and Lemma 2.9 that $m(B \cap \mathcal{G}\pi^{-1}(t^i)) = m(\pi^{-1}(t^i)) m_{t^i}(B)$ for all Borel sets B . Set $\lambda_i = m(\pi^{-1}(t^i))$ and note that

$$m_2(B) = m(B) - \sum_{i=1}^N \lambda_i m_{t^i}(B) = m\left(B \setminus \bigcup_{i=1}^N \mathcal{G}\pi^{-1}(t^i)\right) \geq 0$$

for all Borel sets B . It follows from Lemma 3.1 that m_2 is a harmonic β -KMS measure. This completes the proof because ii) holds by construction. \square

Theorem 5.7. *Assume that G is strongly connected, that $C^*(G)$ is simple and that there are at most countably many exits in G . Let $\beta \in \mathbb{R}$. There are no β -KMS weights unless $\mathbb{P}(-\beta F) \leq 0$. Assume therefore that this holds.*

- 1) *Assume that $A(\beta)$ is recurrent. Then $\mathbb{P}(-\beta F) = 0$ and there is a β -KMS weight, unique up to multiplication by scalars. It is harmonic and the associated measure on $P(G)$ is conservative.*
- 2) *Assume that $A(\beta)$ is transient. The rays of extremal boundary β -KMS weights are in bijective correspondence with the infinite emitters in G , and all extremal harmonic β -KMS weights arise from exit measures of β -summable exits in G .*

Proof. That there are no β -KMS weights unless $\mathbb{P}(-\beta F) \leq 0$ follows from Lemma 4.13. Case 1) follows from Theorem 4.14. The statement concerning boundary β -KMS weights in case 2) follow from Proposition 4.16. In view of Corollary 5.5 it suffices then to show that every non-zero extremal harmonic β -KMS measure m is a multiple of the exit measure m_t of a β -summable exit t . For this note that $P(G)_{\text{wan}} = \pi^{-1}(M)$ where M is the set of exit paths in $P(V)$, cf. (4.3). Since G is a countable graph the set of exit paths tail-equivalent to a given exit path is countable and hence, under the present assumption, the set M is countable. By Theorem 4.10, m is dissipative because $A(\beta)$ is transient, which means that $m(P(G) \setminus \pi^{-1}(M)) = 0$. Therefore, in the decomposition $m = m_1 + m_2$ from Proposition 5.6 the measure m_2 must be zero and by extremality of m there is exactly one exit measure in the decomposition of $m_1 = m$. \square

6. CONSTRUCTING GRAPHS WITH PRESCRIBED STRUCTURE OF KMS WEIGHTS FOR THE GAUGE ACTION

6.1. The setting and the objective. We retain the assumption that G is strongly connected and consists of more than a single loop so that $C^*(G)$ is simple. In addition we assume now that the function F is constant 1 so that α^F is the gauge action on $C^*(G)$. Let A be the adjacency matrix of G , cf. (2.4). Then $A(\beta) = e^{-\beta}A$ and

$$\mathbb{P}(-\beta F) = \mathbb{P}(-\beta) = h(G) - \beta, \quad (6.1)$$

where

$$h(G) = \limsup_n \frac{1}{n} \log A_{vv}^n$$

is independent of which vertex $v \in V$ we consider. The number $h(G)$ is known as *the Gurevich entropy* of G . It follows from Lemma 4.13 that there are no KMS weights at all when $\beta < h(G)$. In the following we will therefore only consider strongly connected graphs G with finite Gurevich entropy. We will call these graphs *admissible*.

Following standard terminology, cf. e.g. [Ru], we will say that G is *recurrent* when

$$\sum_{n=0}^{\infty} A_{vv}^n e^{-nh(G)}$$

is infinite for some and hence all $v \in V$, and that G is *transient* when it is not. In the terminology we use here, G is recurrent or transient exactly when the matrix $e^{-h(G)}A$ is recurrent or transient, respectively.

Let $t = (t_i)_{i=1}^{\infty}$ be an exit path in G and consider $\beta \in \mathbb{R}$. Since we are considering the gauge action we have that $A(\beta) = e^{-\beta}A$ and

$$t^\beta(k) = e^{-(k-1)\beta} t(k),$$

where $t(k) = A_{t_1 t_2} A_{t_2 t_3} \cdots A_{t_{k-1} t_k}$, and

$$t^\beta(k)^{-1} \sum_{n=0}^{\infty} A(\beta)_{vt_k}^n = e^{(k-1)\beta} t(k)^{-1} \sum_{n=0}^{\infty} A_{vt_k}^n e^{-n\beta} \quad (6.2)$$

for all $\beta \in \mathbb{R}$ and all $v \in V$. For a given $\beta \in \mathbb{R}$ the exit path t is β -summable if and only if

$$\sum_{n=0}^{\infty} A_{t_1 t_1}^n e^{-n\beta} < \infty$$

and

$$\lim_{k \rightarrow \infty} e^{(k-1)\beta} t(k)^{-1} \sum_{n=0}^{\infty} A_{t_1 t_k}^n e^{-n\beta} < \infty.$$

The convexity of the exponential function implies therefore that the β -values for which an exit path is β -summable constitutes an interval. When there are L exits, the possible inverse temperatures which the exit measures and their KMS weights can contribute is a set of the form

$$\bigcup_{i=1}^L I_i$$

where each I_i is a sub-interval (open, closed or half-open) in $]h(G), \infty[$ when G is recurrent and in $[h(G), \infty[$ when G is transient. Furthermore, for each $\beta \in \bigcup_{i=1}^L I_i$ the number of extremal rays of β -KMS weights arising from the exits is the number

$$\# \{1 \leq i \leq L : \beta \in I_i\}.$$

We aim now to show that all these possibilities can actually be realized with one necessary restriction in the row-finite case.

6.2. Ruelle graphs. When v is a vertex we let $l_{vv}^k(G)$ be the number of simple loops of length k in G based at v . We introduce now a class of graphs which were considered by Ruelle in Example 2.9 in [Ru].

Definition 6.1. A vertex v in an admissible graph G is a *Ruelle vertex* when

- i) $l_{vv}^1(G) = 1$, and
- ii) $\limsup_n (l_{vv}^n(G))^{\frac{1}{n}} = e^{h(G)}$.

An admissible graph with a Ruelle vertex will be called a *Ruelle graph*.

Lemma 6.2. Let G be an admissible graph and v a vertex in G . Set

$$L = \limsup_n (l_{vv}^n(G))^{\frac{1}{n}}.$$

The following are equivalent:

- i) $L > 0$ and $\sum_{n=1}^{\infty} l_{vv}^n(G) L^{-n} = 1$.
- ii) G is recurrent and $L = e^{h(G)}$.

Proof. Apply Lemma 4.15 to G with $F = 1$ and $\beta = \log L$. □

Proposition 6.3. Let G be an Ruelle graph with Ruelle vertex v . Assume that G is recurrent and let G' be the graph obtained from G by removing the edge e with $s(e) = r(e) = v$. Then G' is transient and $h(G') = h(G)$.

Proof. Since $G' \subseteq G$ we have trivially that $h(G') \leq h(G)$. Let A' be the adjacency matrix of G' . Then $l_{vv}^n(G') \leq (A')_{vv}^n$ and hence

$$\limsup_n (l_{vv}^n(G'))^{\frac{1}{n}} \leq e^{h(G')}.$$

But $l_{vv}^n(G') = l_{vv}^n(G)$ when $n \geq 2$ and v is a Ruelle vertex in G so

$$\limsup_n (l_{vv}^n(G'))^{\frac{1}{n}} = e^{h(G)}.$$

Therefore $h(G') = h(G)$. It follows from Lemma 6.2 that $\sum_{n=1}^{\infty} l_{vv}^n(G) e^{-nh(G)} = 1$ and hence

$$\sum_{n=1}^{\infty} l_{vv}^n(G') e^{-nh(G')} = \sum_{n=2}^{\infty} l_{vv}^n(G) e^{-nh(G)} = 1 - e^{-h(G)} < 1.$$

By Lemma 4.11 this implies that $\sum_{n=0}^{\infty} (A')_{vv}^n e^{-nh(G')} < \infty$, i.e G' is transient. □

6.3. Technical lemmas on series. We need some notation and terminology to describe the intervals of summability for the exits in the examples we are going to construct. Given three sequences $\mathbf{a} = \{a_i\}_{i=1}^\infty$, $\mathbf{b} = \{b_i\}_{i=1}^\infty$ and $\mathbf{c} = \{c_i\}_{i=1}^\infty$, of non-zero natural numbers, let

$$J(\mathbf{a}, \mathbf{b}, \mathbf{c})$$

be the set of real numbers $\beta \in \mathbb{R}$ with the property that

$$\sum_{k=1}^{\infty} \frac{b_k}{a_1 a_2 \cdots a_k} e^{-k\beta} < \infty. \quad (6.3)$$

and

$$\sum_{k=1}^{\infty} \frac{c_k}{a_1 a_2 \cdots a_k} e^{k\beta} < \infty. \quad (6.4)$$

Lemma 6.4. *Let $0 < r \leq R$ be real numbers, and let I be one of the intervals*

$$[r, R],]r, R], [r, R[,]r, R[, [r, \infty[,]r, \infty[.$$

There is a choice of sequences $\mathbf{a} = \{a_i\}_{i=1}^\infty$, $\mathbf{b} = \{b_i\}_{i=1}^\infty$ and $\mathbf{c} = \{c_i\}_{i=1}^\infty$ of non-zero natural numbers such that $I = J(\mathbf{a}, \mathbf{b}, \mathbf{c})$.

For the proof of Lemma 6.4 we need the following observation.

Lemma 6.5. *Let $\{q_n\}_{n=1}^\infty, \{q'_n\}_{n=1}^\infty$ be sequences of positive rational numbers. There are sequences $\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty, \{c_n\}_{n=1}^\infty$ of non-zero natural numbers such that*

$$q_n = \frac{b_n}{a_1 a_2 \cdots a_n} \quad \text{and} \quad q'_n = \frac{c_n}{a_1 a_2 \cdots a_n}$$

for all n .

Proof. Left to the reader. □

Proof of Lemma 6.4: We give only the details in the proof in the cases $I =]r, R[$ and $I =]r, R]$. The remaining four cases can be handled in a similar way. Set $S = e^R$ and $s = e^{-r}$. Let $\{\epsilon_n\}$ and $\{\epsilon'_n\}$ be sequences of positive real numbers and $\{q_n\}, \{q'_n\}$ sequences of positive rational numbers such that

$$\sum_{n=1}^{\infty} \epsilon_n s^n < \infty, \quad \sum_{n=1}^{\infty} \epsilon'_n S^n < \infty,$$

and

$$\frac{1}{(s + n^{-1})^n} \leq \frac{1}{s^n} - \epsilon_n \leq q_n \leq \frac{1}{s^n}, \quad \frac{1}{(S + n^{-1})^n} \leq \frac{1}{S^n} - \epsilon'_n \leq q'_n \leq \frac{1}{S^n}$$

for all n . The radii of convergence of the powers series $\sum_{n=1}^{\infty} q_n z^n$ and $\sum_{n=1}^{\infty} q'_n z^n$ are s and S , respectively. Note that

$$\sum_{n=1}^{\infty} q_n s^n \geq \sum_{n=1}^{\infty} \left(\frac{1}{s^n} - \epsilon_n \right) s^n = \infty.$$

Similar, $\sum_{n=1}^{\infty} q'_n S^n = \infty$. It follows that if we let $\mathbf{a} = \{a_i\}_{i=1}^\infty$, $\mathbf{b} = \{b_i\}_{i=1}^\infty$ and $\mathbf{c} = \{c_i\}_{i=1}^\infty$ be the sequences obtained by applying Lemma 6.5 to the sequence $\{q_n\}$ and $\{q'_n\}$, we have that $]r, R[= J(\mathbf{a}, \mathbf{b}, \mathbf{c})$.

To handle the case $I =]r, R]$ we proceed as follows: We use Lemma 6.5 to find sequences $\mathbf{a} = \{a_i\}_{i=1}^\infty$, $\mathbf{b} = \{b_i\}_{i=1}^\infty$ and $\mathbf{c} = \{c_i\}_{i=1}^\infty$ such that

$$q_n = \frac{b_n}{a_1 a_2 \cdots a_n},$$

and

$$\frac{q'_n}{n^2} = \frac{c_n}{a_1 a_2 \cdots a_n}.$$

The series $\sum_{n=1}^\infty \frac{q'_n}{n^2} z^n$ has S as its radius of convergence, and

$$\sum_{n=1}^\infty \frac{q'_n}{n^2} S^n \leq \sum_{n=1}^\infty \frac{1}{n^2} < \infty.$$

It follows that (6.3) holds if and only if $e^{-\beta} < s$ while (6.4) holds if and only if $e^\beta \leq S$. Hence $J(\mathbf{a}, \mathbf{b}, \mathbf{c}) =]r, R]$. \square

Lemma 6.6. *Let $D \in]0, 1[$ and let $\{a_n\}_{n=2}^\infty$ be a sequence of non-negative integers such that*

$$\sum_{n=2}^\infty a_n D^n < 1 - D. \quad (6.5)$$

There is a sequence $\{b_n\}_{n=1}^\infty$ of non-negative integers such that

- i) $b_1 = 1$,
- ii) $b_n \geq a_n$ for all $n \geq 2$,
- iii) $b_n - a_n \geq 2$ for infinitely many n ,
- iv) $\limsup_n (b_n)^{\frac{1}{n}} = D^{-1}$ and
- v) $\sum_{n=1}^\infty b_n D^n = 1$.

Proof. Set $s = 1 - D - \sum_{n=2}^\infty a_n D^n$ and choose a sequence of real numbers $0 < s_1 < s_2 < s_3 < \cdots$ such that $\lim_{m \rightarrow \infty} s_m = s$. Choose first a sequence $\{r_n\}_{n=1}^\infty$ in \mathbb{N} such that

$$\lim_{m \rightarrow \infty} (s_m - s_{m-1})^{\frac{1}{r_m}} = 1. \quad (6.6)$$

We will inductively construct sequences of natural numbers $\{k_m\}_{m=1}^\infty$ and $\{n_m\}_{m=1}^\infty$ such that $\{n_m\}$ is strictly increasing and

- a) $n_m \geq r_m$ for all $m \geq 1$,
- b) $k_m D^{n_m} \geq s_m - s_{m-1}$ for all $m \geq 2$, and
- c) $0 < s_m - \sum_{i=1}^m k_i D^{n_i} \leq \frac{s}{m}$ for all $m \geq 1$.

To start the induction note that since $\lim_{n \rightarrow \infty} D^n = 0$, we can choose $n_1, k_1 \in \mathbb{N}$ such that $n_1 \geq \max\{r_1, 2\}$ and $k_1 D^{n_1} < s_1$. Assume then that n_i, k_i have been found when $i \leq m-1$. To simplify notation, set

$$\Sigma_{m-1} = \sum_{i=1}^{m-1} k_i D^{n_i}.$$

Note that

$$\begin{aligned} s_m - s_{m-1} + \frac{m-1}{m} (s_{m-1} - \Sigma_{m-1}) \\ = s_m - \Sigma_{m-1} - \frac{1}{m} (s_{m-1} - \Sigma_{m-1}) < s_m - \Sigma_{m-1}. \end{aligned} \quad (6.7)$$

Using that $\lim_{n \rightarrow \infty} D^n = 0$ we find an $n_m > \max\{r_m, n_{m-1}\}$ in \mathbb{N} such that

$$3D^{n_m} \leq \frac{1}{m}(s_{m-1} - \Sigma_{m-1}). \quad (6.8)$$

Let k_m be the least natural number such that

$$k_m D^{n_m} > s_m - s_{m-1} + \frac{m-1}{m}(s_{m-1} - \Sigma_{m-1}). \quad (6.9)$$

Then b) certainly holds, and since k_m is the least natural number with the stated property it follows by use of (6.8) and (6.7) that

$$k_m D^{n_m} < s_m - \Sigma_{m-1},$$

giving us the strict inequality in c). The other inequality in c) follows because

$$s_m - \Sigma_{m-1} - \frac{1}{m}(s_{m-1} - \Sigma_{m-1}) < k_m D^{n_m} < s_m - \Sigma_{m-1}$$

by construction.

Set $c_n = k_m$ when $n = n_m$ and $c_n = 0$ when $n \notin \{n_k : k \in \mathbb{N}\}$, and then $b_n = c_n + a_n$, $n \geq 2$, and $b_1 = 1$. Then i), ii) and iii) hold and v) follows from c):

$$\sum_{n=1}^{\infty} b_n D^n = D + \sum_{n=2}^{\infty} a_n D^n + \sum_{m=1}^{\infty} k_m D^{n_m} = D + \sum_{n=2}^{\infty} a_n D^n + s = 1.$$

To see that iv) holds note first that v) implies that $\limsup_n (b_n)^{\frac{1}{n}} \leq D^{-1}$. To establish the reversed inequality note that b) implies that

$$\limsup_m (k_m)^{\frac{1}{n_m}} D = \limsup_m (k_m D^{n_m})^{\frac{1}{n_m}} \geq \limsup_m (s_m - s_{m-1})^{\frac{1}{n_m}}.$$

Thanks to (6.6) and because $n_m \geq r_m$, it follows that $\limsup_m (s_m - s_{m-1})^{\frac{1}{n_m}} = 1$ and hence that $\limsup_m (k_m)^{\frac{1}{n_m}} \geq D^{-1}$. This finishes the proof because

$$\limsup_n (b_n)^{\frac{1}{n}} \geq \limsup_n (c_n)^{\frac{1}{n}} = \limsup_m (k_m)^{\frac{1}{n_m}}.$$

□

7. THE ROW-FINITE CASES

In this section we prove the following two theorems.

Theorem 7.1. *Let $h \in]0, \infty[$ be a positive real number. Let \mathbb{I} be a finite or countably infinite collection of intervals in $]h, \infty[$. Assume that*

$$I_0 \in]h, \infty[$$

for at least one $I_0 \in \mathbb{I}$.

There is a strongly connected recurrent row-finite graph G with Gurevich entropy $h(G) = h$ such that the set of exits in G is in bijective correspondence with \mathbb{I} . Furthermore, for $\beta \geq h$ there are the following extremal β -KMS weights for the gauge action on $C^(G)$:*

- *For $\beta > h$ the rays of extremal β -KMS weights are in bijective correspondence with the set*

$$\{I \in \mathbb{I} : \beta \in I\}.$$

- *For $\beta = h$ there is a unique ray of extremal h -KMS weights.*

Theorem 7.2. *Let $h \in]0, \infty[$ be a positive real number. Let \mathbb{I} be a finite or countably infinite collection of intervals in $[h, \infty[$. Assume that*

$$I_0 = [h, \infty[$$

for at least one $I_0 \in \mathbb{I}$.

There is a strongly connected transient row-finite graph G with Gurevich entropy $h(G) = h$ such that the set of exits in G is in bijective correspondence with \mathbb{I} . Furthermore, for $\beta \geq h$ the rays of extremal β -KMS weights for the gauge action on $C^(G)$ are in bijective correspondence with the set*

$$\{I \in \mathbb{I} : \beta \in I\}.$$

In order to fully appreciate these results it must be observed that the assumed presence of an interval in \mathbb{I} of maximal size is not accidental. To explain this we introduce a key ingredient in the following constructions. An exit path $(v_i)_{i=1}^\infty$ in G is a *bare exit path* when $s(r^{-1}(v_{i+1})) = v_i$ for all i . It is easy to see that a bare exit is β -summable for all β in the largest possible interval, cf. Lemma 7.3, namely for all β such that $e^{-\beta}A$ is transient, where A denotes the adjacency matrix of the graph. And it is a fact, proved in Appendix 9, that a strongly connected graph with infinitely many vertexes and at most countably many exits must have a bare exit when it is row-finite. So the presence of a largest possible interval among the intervals in \mathbb{I} can not be avoided; for strongly connected row-finite graphs with infinitely many vertexes and at most countably many exits there must be an exit which contributes a β -KMS weight for the gauge action for all β in the largest possible interval.

Lemma 7.3. *Assume that G is strongly connected with adjacency matrix A . A bare exit path $t = (t_i)_{i=1}^\infty$ is β -summable if and only if $\sum_{n=0}^\infty A_{t_1 t_1}^n e^{-n\beta} < \infty$.*

Proof. Assuming that $\sum_{n=0}^\infty A_{t_1 t_1}^n e^{-n\beta} < \infty$ we observe that

$$\sum_{n=0}^\infty A_{t_1 t_k}^n e^{-n\beta} = t(k) e^{-(k-1)\beta} \sum_{n=0}^\infty A_{t_1 t_1}^n e^{-n\beta}$$

since t is bare. Hence

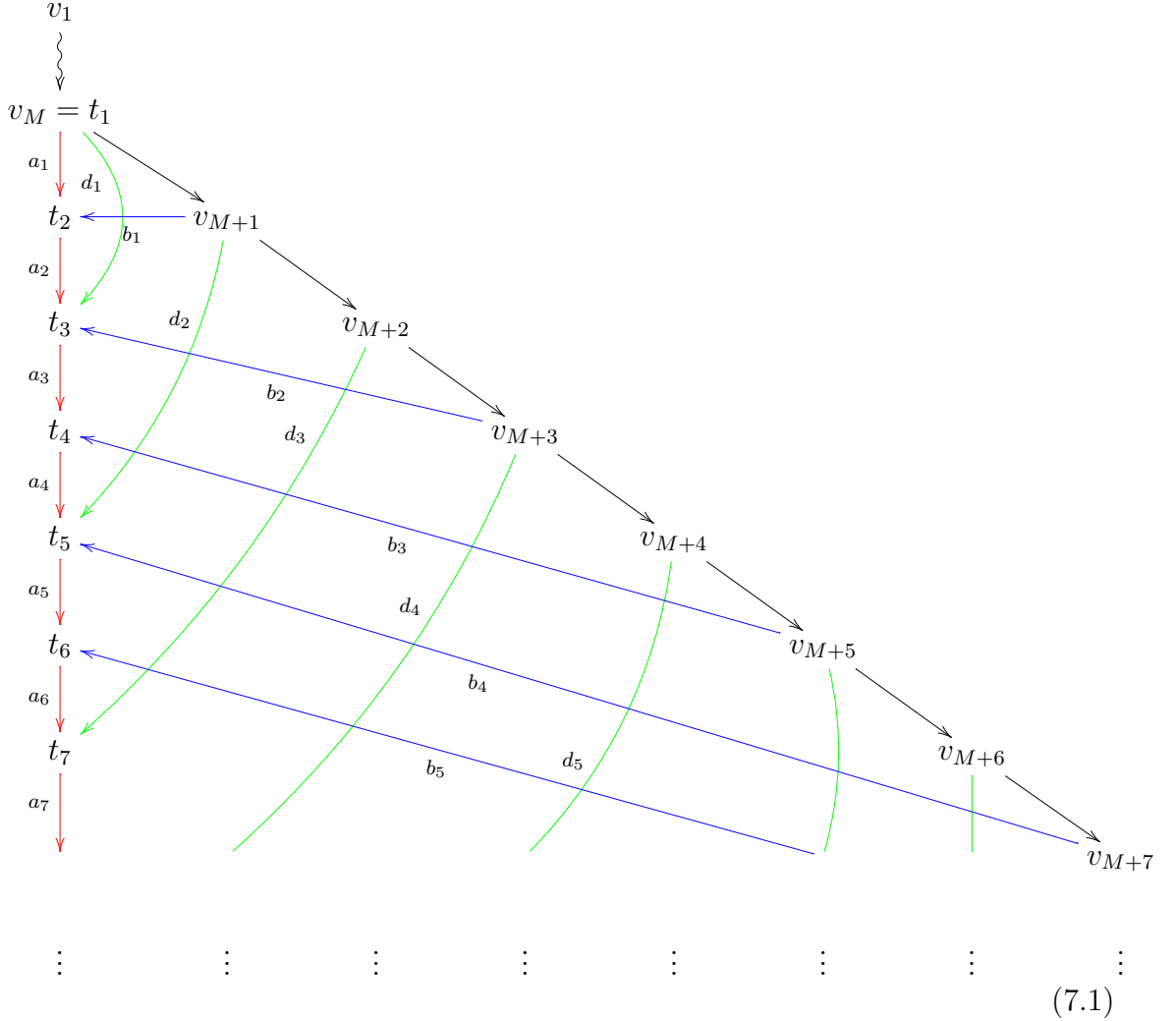
$$e^{(k-1)\beta} t(k)^{-1} \sum_{n=0}^\infty A_{t_1 t_k}^n e^{-n\beta} = \sum_{n=0}^\infty A_{t_1 t_1}^n e^{-n\beta} < \infty$$

for all k , and t is β -summable. The converse is trivial. □

7.1. The basic construction I: Adding an exit. In the following construction of examples a key step is the attachment of an exit path to an existing graph by a procedure which depends on whether the resulting graph should be row-finite or not. In this section we describe the relevant method in the row-finite case.

Let G_0 be an arbitrary directed graph. We assume that $(v_i)_{i=1}^\infty$ is a bare exit in G_0 . Let I be an arbitrary interval in $]0, \infty[$ and choose with the aid of Lemma 6.5 three sequences $\mathbf{a} = (a_i)_{i=1}^\infty$, $\mathbf{b} = (b_i)_{i=1}^\infty$, $\mathbf{c} = (c_i)_{i=1}^\infty$ of non-zero natural numbers such that $J(\mathbf{a}, \mathbf{b}, \mathbf{c}) = I$. For each $i \geq 1$ we let $d_i = c_i a_{i+1} a_{i+2} \cdots a_{2i}$. Add to G_0 the vertexes t_2, t_3, t_4, \dots and set $t_1 = v_M$, where $M \in \mathbb{N}$ is arbitrary. Add a_i arrows from t_i to t_{i+1} , add d_i arrows from v_{M+i-1} to t_{2i+1} and b_i arrows from v_{M-1+2i} to

t_{i+1} for $i = 1, 2, 3, \dots$. Let G denote the resulting graph. If we ignore the part of $G_0 \setminus \{v_i\}_{i=1}^\infty$ with no contact to the new vertexes, the graph G looks at follows.



The vertex v_M will be referred to as *the first contact vertex*.

7.2. Proof of Theorems 7.1 and 7.2: Let I_1, I_2, I_3, \dots be a numbering of the intervals in $\mathbb{I} \setminus \{I_0\}$. Start with the following graph H_0 :

$$v_1 \longrightarrow v_2 \longrightarrow v_3 \longrightarrow v_4 \longrightarrow v_5 \longrightarrow \dots \quad (7.2)$$

the vertexes of which is a bare exit path t^0 . Using this exit path as a backbone we shall build a recurrent Ruelle graph G with v_1 as a Ruelle vertex such that G has the properties described in Theorem 7.1 while the graph G' obtained from G by removing the edge starting and ending at v_1 will be a graph with the properties described in Theorem 7.2.

Attach to H_0 an exit path t^1 as described in Section 7.1 using v_1 as first contact vertex and using multiplicities $\mathbf{a}(1) = \{a_i\}$, $\mathbf{b}(1) = \{b_i\}$ and $\mathbf{c}(1) = \{c_i\}$ such that $J(\mathbf{a}(1), \mathbf{b}(1), \mathbf{c}(1)) = I_1$. Let H_1 be the resulting graph. Attach to H_1 an exit path t^2 using v_2 as first contact vertex and using multiplicities $\mathbf{a}(2)$, $\mathbf{b}(2)$ and $\mathbf{c}(2)$ such that $J(\mathbf{a}(2), \mathbf{b}(2), \mathbf{c}(2)) = I_2$. Let H_2 be the resulting graph and continue

recursively, either indefinitely or until the intervals in $\mathbb{I} \setminus \{I_0\}$ have been used up. Set

$$H = \bigcup_i H_i.$$

Since we used different first contact vertexes the graph H will be row-finite. In case I_0 is the only interval in \mathbb{I} we haven't done anything and the graph H is just the bare exit H_0 we started with. In any case, H is certainly not strongly connected.

We next add to H a sequence $\mu_i, i = 1, 2, 3, \dots$, paths going from vertexes in the exit paths we have attached to the vertex v_1 , in order to make the resulting graph G_0 strongly connected. The paths are mutually disjoint, both from each other and from H , except for their initial and terminal vertexes. We want also to arrange that

$$\sum_{n=1}^{\infty} l_{v_1 v_1}^n(G_0) e^{-nh} < 1 - e^{-h}. \quad (7.3)$$

This can be done in many ways; the following procedure is perhaps slightly brutal, but it is easy to describe. In the following a *simple path* from v to w in a graph K is a finite path $\mu = e_1 e_2 \dots e_n \in E^n$ in $P_f(K)$ such that $s(e_1) = v$, $r(e_n) = w$, $s(e_j) \neq v \forall j \in \{2, 3, \dots, n\}$. Let w_1, w_2, w_3, \dots be a numbering of the vertexes in the attached exit paths t^i , $i \geq 1$. Observe that for each i there are only finitely many, say a_i , simple paths in H from v_1 to w_i . Let l_i be the maximal length of such a path. Choose $n_1 \in \mathbb{N}$ so large that $a_1 \leq e^{n_1 \frac{h}{2}}$ and $\sum_{j \geq n_1} e^{-\frac{jh}{2}} < 1 - e^{-h}$, and then recursively $n_{k+1} > n_k + l_k$ such that

$$a_{k+1} \leq e^{\frac{n_{k+1}h}{2}}.$$

Add to H for each i a path μ_i from w_i to v_1 of length n_i . Since every simple loop in G_0 from v_1 back to itself must pass through a w_i , we find that $l_{v_1 v_1}^n(G_0) = 0$ unless $n \in \bigcup_k [n_k, n_k + l_k]$ and hence

$$\begin{aligned} \sum_{n=1}^{\infty} l_{v_1 v_1}^n(G_0) e^{-nh} &= \sum_{k=1}^{\infty} \sum_{j=n_k}^{n_k+l_k} l_{v_1 v_1}^j(G_0) e^{-jh} \leq \sum_{k=1}^{\infty} \sum_{j=n_k}^{n_k+l_k} l_{v_1 v_1}^j(G_0) e^{-n_k h} \\ &= \sum_{k=1}^{\infty} a_k e^{-n_k h} \leq \sum_{k=1}^{\infty} e^{-\frac{n_k h}{2}} \leq \sum_{j \geq n_1} e^{-\frac{jh}{2}} < 1 - e^{-h}, \end{aligned} \quad (7.4)$$

as desired. To make the final adjustment to the graph we first use Lemma 6.6 to get a sequence $\{b_n\}_{n=1}^{\infty}$ of non-negative integers such that $b_1 = 1$, $b_n \geq l_{v_1 v_1}^n(G_0)$ for all n , $\limsup_n (b_n)^{\frac{1}{n}} = e^h$ and

$$\sum_{n=1}^{\infty} b_n e^{-nh} = 1.$$

For each $n \in \mathbb{N}$ we add $b_n - l_{v_1 v_1}^n(G_0)$ arrows to G_0 going from v_n to v_1 . The resulting graph G is then a recurrent Ruelle graph with Gurevics entropy $h(G) = h$ by Lemma 6.2. By Proposition 6.3 we obtain a transient graph G' with the same entropy by removing the single edge going from v_1 to v_1 . Both graphs are row-finite. We claim G will have the properties described in Theorem 7.1 and G' the properties described in Theorem 7.2. For both graphs it is easy to see that the exits correspond bijectively to the exit paths t^i , $i \geq 0$. In view of Theorem 5.7 it suffices therefore to show that each exit path t^i will be β -summable iff $\beta \in I_i$ and $e^{-\beta} A$ is transient,

where A is the adjacency matrix of the graph in question, i.e. either G or G' . The argument for this is a matter of book-keeping and runs as follows.

For the bare exit the claim follows from Lemma 7.3 so we consider one of the attached exit paths, say $t = t^M$ so that the graph (7.1) depicts the relevant part of G . For $k \geq 2$, let $\mathbb{L}_k(n)$ denote the set of simple paths from t_1 to t_k of length n , and set

$$\mathbb{L}_k = \bigcup_n \mathbb{L}_k(n).$$

Let \mathbb{A}_k be the set of all finite paths $\mu \in P_f(G)$ such that $s(\mu) = t_1$, $r(\mu) = t_k$. Finally, we let \mathbb{B} be the set of all finite paths (loops) $\mu \in P_f(G)$ such that $s(\mu) = r(\mu) = t_1$. Then

$$\mathbb{A}_k = \sqcup_{\mu' \in \mathbb{L}_k} \{\mu'' \mu' : \mu'' \in \mathbb{B}\}.$$

It follows that

$$\sum_{n=0}^{\infty} A_{t_1 t_k}^n e^{-n\beta} = \sum_{\mu \in \mathbb{A}_k} e^{-|\mu|\beta} = \alpha \sum_{\mu \in \mathbb{L}_k} e^{-|\mu|\beta}$$

where

$$\alpha := \sum_{n=0}^{\infty} A_{t_1 t_1}^n e^{-n\beta}.$$

Set

$$x_k = e^{(k-1)\beta} t(k)^{-1} \sum_{n=1}^{k-1} e^{-n\beta} \# \mathbb{L}_k(n) \quad (7.5)$$

and

$$y_k = e^{(k-1)\beta} t(k)^{-1} \sum_{n \geq k} e^{-n\beta} \# \mathbb{L}_k(n). \quad (7.6)$$

Then

$$e^{(k-1)\beta} t(k)^{-1} \sum_{n=0}^{\infty} A_{t_1 t_k}^n e^{-n\beta} = \alpha(x_k + y_k),$$

and t will be a β -summable exit path iff $\alpha < \infty$ and $\sup_k x_k$ and $\sup_k y_k$ are both finite, or alternatively, since $x_k + y_k$ increases with k , iff $\alpha < \infty$ and $\sup_k x_{2k+1}$ and $\sup_k y_{2k+1}$ are both finite. A simple count shows that

$$\sum_{n=1}^{2k} e^{-n\beta} \# \mathbb{L}_{2k+1}(n) = e^{-2k\beta} a_1 a_2 \cdots a_{2k} + e^{-k\beta} d_k + \sum_{i=1}^{k-1} e^{-(2k-i)\beta} d_i a_{2i+1} a_{2i+2} \cdots a_{2k}, \quad (7.7)$$

for $k \geq 1$, while

$$\sum_{n \geq k} e^{-n\beta} \# \mathbb{L}_k(n) = e^{-(2k-2)\beta} b_{k-1} + \sum_{i=1}^{k-2} e^{-(k+i-1)\beta} b_i a_{i+1} a_{i+2} \cdots a_{k-1}, \quad (7.8)$$

for $k \geq 2$. Inserting (7.7) and (7.8) into (7.5) and (7.6), and remembering that $d_i = c_i a_{i+1} \cdots a_{2i}$, we find that

$$x_{2k+1} = 1 + \sum_{i=1}^k e^{i\beta} c_i (a_1 a_2 \cdots a_i)^{-1},$$

for $k \geq 1$, and

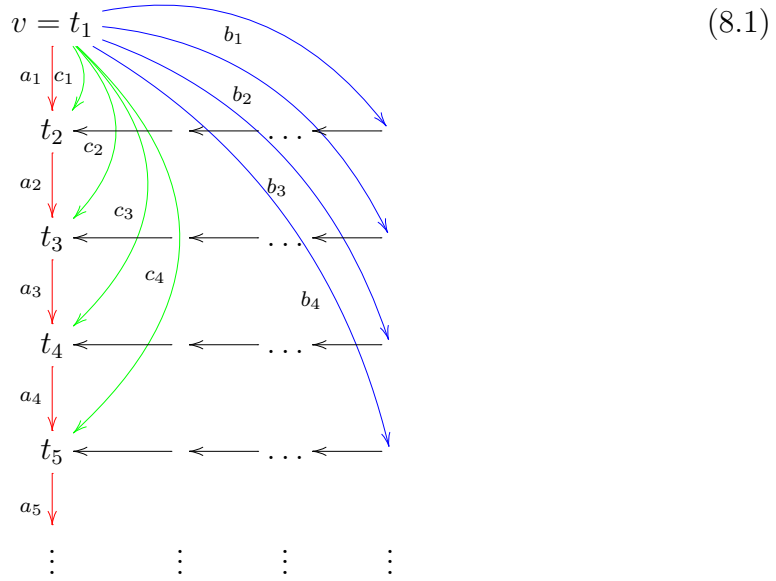
$$y_k = \sum_{i=1}^{k-1} e^{-i\beta} b_i (a_1 a_2 \cdots a_i)^{-1},$$

for $k \geq 2$. It follows that $\sup_k x_{2k+1}$ and $\sup_k y_{2k+1}$ will both be finite if and only if $\beta \in J(\mathbf{a}, \mathbf{b}, \mathbf{c}) = I_M$. \square

8. GRAPHS WITH INFINITE EMITTERS

In this section we prove the two Theorems 1.1 and 1.2 from the introduction.

8.1. The basic construction II: Adding an exit. Let G_0 be an arbitrary graph and v a vertex in G_0 . In analogy with what we did in Section 7.1 we want to add to G_0 a new exit, but this time the construction will make v an infinite emitter, also when it wasn't in G_0 . Let I an arbitrary interval in $]0, \infty[$ and choose with the aid of Lemma 6.4 three sequences $\mathbf{a} = (a_i)_{i=1}^\infty$, $\mathbf{b} = (b_i)_{i=1}^\infty$, $\mathbf{c} = (c_i)_{i=1}^\infty$ of non-zero natural numbers such that $J(\mathbf{a}, \mathbf{b}, \mathbf{c}) = I$. Set $t_1 = v$ and add to G_0 the vertexes t_2, t_3, t_4, \dots . Also, add a_i arrows from t_i to t_{i+1} , $i \geq 1$, add c_i arrows from t_1 to t_{i+1} , $i \geq 1$, and add a path of length $2i$ from t_1 to t_i , $i \geq 2$, whose first edge has multiplicity b_{i-1} and the rest multiplicity 1. Then, if we ignore the part of G_0 with no contact to the new vertexes, the graph G looks at follows.



The labels show the multiplicity of the edge; unlabeled black edges have multiplicity 1.

8.2. Proof of Theorem 1.1 and Theorem 1.2: The fundamental idea of the proof is the same as for the proof of Theorems 7.1 and 7.2: We construct a recurrent Ruelle graph G with the properties described in Theorem 1.1 and then remove a single edge to obtain a graph with the properties described in Theorem 1.2. Let M be a set which is countably infinite if $N = \infty$ or else has $N - 1$ elements, and let I_1, I_2, I_3, \dots be a numbering of the intervals in \mathbb{I} . Let H_0 be the graph which only consists of the vertex v and for each $i \in M$ a vertex w_i , and an arrow from v to w_i . Attach to H_0 an exit path t^1 whose first vertex is v as described in

Section 8.1 using multiplicities $\mathbf{a}(1) = \{a_i\}$, $\mathbf{b}(1) = \{b_i\}$ and $\mathbf{c}(1) = \{c_i\}$ such that $J(\mathbf{a}(1), \mathbf{b}(1), \mathbf{c}(1)) = I_1$. Let H_1 be the resulting graph. Attach to H_1 an exit path t^2 using multiplicities $\mathbf{a}(2)$, $\mathbf{b}(2)$ and $\mathbf{c}(2)$ such that $J(\mathbf{a}(2), \mathbf{b}(2), \mathbf{c}(2)) = I_2$. Let H_2 be the resulting graph and continue recursively, either indefinitely or until the intervals in \mathbb{I} have been used up. Set

$$H = \bigcup_i H_i.$$

We next add to H a sequence $\mu_i, i = 1, 2, 3, \dots$, of paths going from the vertexes in the exit paths we have attached to the vertex v . The paths are mutually disjoint, both from each other and from H , except for their initial and terminal vertexes. We want also to arrange that in the resulting graph G_0 we have that $l_{vv}^1(G_0) = 0$ and

$$\sum_{n=1}^{\infty} l_{vv}^n(G_0) e^{-nh} < 1 - e^{-h}. \quad (8.2)$$

This can be done in exactly the same way as in the proof in Section 7.1, but note that this time G_0 is still not strongly connected unless $N = 1$ since the vertexes $w_i, i \in M$, are sinks. To make the final adjustment to the graph we first use Lemma 6.6 to get a sequence $\{b_n\}_{n=1}^{\infty}$ of non-negative integers such that $b_1 = 1$, $b_n \geq l_{vv}^n(G_0)$ for all n , $\limsup_n (b_n)^{\frac{1}{n}} = e^h$ and

$$\sum_{n=1}^{\infty} b_n e^{-nh} = 1.$$

By condition iii) in Lemma 6.6,

$$W = \{n \geq 2 : b_n - l_{vv}^n(G_0) > 0\}$$

is infinite. When $N = 1$ so that $M = \emptyset$, we add $b_n - l_{vv}^n(G)$ loops of length n from v back to v for each $n \in W$, all mutually disjoint except at v . When $N \geq 2$ we partition W into $\#M$ sets $W_i, i \in M$, with infinitely many elements in each set. For each $k \in W_i$ we then add $b_k - l_{vv}^k(G_0)$ paths of length $k - 1$ from w_i to v . The paths must be mutually disjoint and disjoint from G_0 except at their initial and terminal vertex. Finally, add a single edge from v back to itself. In the resulting graph all $w_i, i \in M$, are now infinite emitters, and

$$\sum_{n=1}^{\infty} l_{vv}^n(G) e^{-nh} = 1.$$

It follows from Lemma 6.2 that G is a recurrent Ruelle graph with Gurevics entropy $h(G) = h$. By Proposition 6.3 we obtain a transient graph G' with the same entropy by removing the single edge going from v to v . To check that G has the properties described in Theorem 1.1 and G' the properties described in Theorem 1.2 it remains only to check the assertions regarding the exits. For both graphs it is easy to see that the exits correspond bijectively to the exit paths we have added. In view of Theorem 5.7 it suffices therefore to show that the exit t corresponding to one of the intervals in \mathbb{I} , say I_i , will be β -summable iff $\beta \in I_i$ and $e^{-\beta}A$ is transient, where A is the adjacency matrix of the graph in question, i.e. either G or G' . Let $\mathbf{a} = (a_i)_{i=1}^{\infty}$, $\mathbf{b} = (b_i)_{i=1}^{\infty}$, $\mathbf{c} = (c_i)_{i=1}^{\infty}$ be the three sequences of non-zero natural numbers used to construct H_i from H_{i-1} and having the property that $J(\mathbf{a}, \mathbf{b}, \mathbf{c}) = I_i$.

Using the same notation as in the proof of Theorems 7.1 and 7.2 we find this time that

$$\begin{aligned} \sum_{n=1}^k e^{-n\beta} \#\mathbb{L}_{k+1}(n) &= c_k e^{-\beta} + a_k c_{k-1} e^{-2\beta} + a_k a_{k-1} c_{k-2} e^{-3\beta} + \\ &\quad \cdots + a_k a_{k-1} \cdots a_2 c_1 e^{-k\beta} + a_k a_{k-1} a_{k-2} \cdots a_1 e^{-k\beta}. \end{aligned}$$

Hence

$$x_{k+1} = e^{k\beta} t(k+1)^{-1} \sum_{n=1}^k e^{-n\beta} \#\mathbb{L}_{k+1}(n) = c_1^{-1} \left(1 + e^{-\beta} \sum_{i=1}^k \frac{c_i}{a_1 a_2 \cdots a_i} e^{i\beta} \right).$$

We see that the limit $\lim_{k \rightarrow \infty} x_k$ is finite if and only if (6.4) holds. Similarly we find that

$$\begin{aligned} \sum_{n \geq k+1} e^{-n\beta} \#\mathbb{L}_{k+1}(n) &= a_k a_{k-1} \cdots a_2 b_1 e^{-(3+k)\beta} + a_k a_{k-1} \cdots a_3 b_2 e^{-(4+k)\beta} + \\ &\quad \cdots + a_k b_{k-1} e^{-(2k+1)\beta} + b_k e^{-(2k+2)\beta}. \end{aligned}$$

It follows that

$$\begin{aligned} c_1 y_{k+1} &= c_1 e^{k\beta} t(k+1)^{-1} \sum_{n \geq k+1} e^{-n\beta} \#\mathbb{L}_{k+1}(n) \\ &= a_1^{-1} b_1 e^{-3\beta} + (a_2 a_1)^{-1} b_2 e^{-4\beta} + \\ &\quad \cdots + (a_{k-1} a_{k-2} \cdots a_1)^{-1} b_{k-1} e^{-(k+1)\beta} + (a_k a_{k-1} \cdots a_1)^{-1} b_k e^{-(k+2)\beta}. \end{aligned}$$

Thus $\lim_{k \rightarrow \infty} y_k$ is finite if and only if (6.3) holds. Since

$$e^{(k-1)\beta} t(k)^{-1} \sum_{n=0}^{\infty} A_{vt_k}^n e^{-n\beta} = \alpha(x_k + y_k),$$

where $\alpha = \sum_{n=0}^{\infty} A_{vv}^n e^{-n\beta}$ we find that t is β -summable if and only if $e^{-\beta} A$ is transient and $\beta \in J(\mathbf{a}, \mathbf{b}, \mathbf{c}) = I_i$. □

9. APPENDIX

An exit path $(v_i)_{i=1}^{\infty}$ is *bare* when

$$\#s(r^{-1}(v_{i+1})) = 1 \tag{9.1}$$

for all i , and *eventually bare* when (9.1) holds for all i large enough. An exit is *bare* when one of its representing exit paths is bare, in which case they are all eventually bare. The purpose with this Appendix is to prove

Theorem 9.1. *Let G be a strongly connected row-finite graph with infinitely many vertexes and at most countably many exits. Then G contains a bare exit.*

The first step is to show that there exist exit paths.

Lemma 9.2. *Assume that G is a strongly connected row-finite graph with infinitely many vertexes. Then G contains an exit path.*

Proof. This was proved by Van Cyr in his thesis, cf. page 94 in [Cy]. Here is the argument: Let $v_0, v_1, v_2, v_3, \dots$ be a numbering of the vertexes in V . For each i choose a finite path μ_i from v_0 to v_i of minimal length. Since $s^{-1}(v_0)$ is finite there are infinitely many μ_i 's that share the first edge, e_1 say. Among them there are infinitely many that share the second edge, e_2 , and so on. This results in an infinite path $p = e_1 e_2 e_3 e_4 \dots$ in which the vertexes only occur once. It follows that the vertexes in p form an exit path. \square

When $\mu = e_1 e_2 \dots e_n$ is a finite path in G and $F \subseteq V$ is a set of vertexes, we write $\mu \cap F = \emptyset$ when μ does not contain any vertex from F , i.e.

$$\bigcup_{i=1}^n \{s(e_i), r(e_i)\} \cap F = \emptyset.$$

Let $(v_i)_{i=1}^\infty$ and $(w_i)_{i=1}^\infty$ be exit paths in G . We write

$$(v_i)_{i=1}^\infty \leq (w_i)_{i=1}^\infty$$

when the following holds: For every finite subset $F \subseteq V$ and every $N \in \mathbb{N}$ there are $n, m \geq N$ and a finite path μ in G such that $s(\mu) = v_n$, $r(\mu) = w_m$ and $\mu \cap F = \emptyset$. Given two exits t and t' in G we write $t \leq t'$ when there are exit paths, $(v_i)_{i=1}^\infty$ and $(w_i)_{i=1}^\infty$, representing t and t' , respectively, such that $(v_i)_{i=1}^\infty \leq (w_i)_{i=1}^\infty$.

Let \mathcal{E} be the collection of exits in G .

Lemma 9.3. *Let G be a strongly connected graph with infinitely many vertexes and at most countably many exits. \mathcal{E} is partially ordered by the relation \leq .*

Proof. Only the anti-symmetry condition is not obvious. For this we must show that when $(v_i)_{i=1}^\infty$ and $(w_i)_{i=1}^\infty$ are exit paths such that

$$(v_i)_{i=1}^\infty \leq (w_i)_{i=1}^\infty \leq (v_i)_{i=1}^\infty, \quad (9.2)$$

then the paths are tail-equivalent. It follows from (9.2) that we can go back and forth between $(v_i)_{i=1}^\infty$ and $(w_i)_{i=1}^\infty$ with paths that eventually leave all finite subsets of vertexes. In particular, we can find sequences $n_1 < n_2 < \dots$ and $m_1 < m_2 < \dots$ in \mathbb{N} and elements $\mu_i, \nu_i \in \bigcup_n V^n$, $i = 1, 2, 3, \dots$, such that

$$z = v_{n_1} \mu_1 w_{m_1} \nu_1 v_{n_2} \mu_2 w_{m_2} \nu_2 \dots$$

is an exit path in G with the property that v_{n_i} does not occur in $\mu_i w_{m_i} \nu_i v_{n_{i+1}} \mu_{i+1} w_{m_{i+1}} \dots$ and w_{m_i} does not occur in $\nu_i v_{n_{i+1}} \mu_{i+1} w_{m_{i+1}} \nu_{i+1} \dots$. For each $i \in \mathbb{N}$, set $a_i^0 = v_{n_i} v_{n_{i+1}} \dots v_{n_{i+1}-1}$ and $a_i^1 = v_{n_i} \mu_i w_{m_i} \nu_i$. For each $(i_j)_{j=1}^\infty \in \{0, 1\}^\mathbb{N}$ we can then consider the exit path

$$P((i_j)_{j=1}^\infty) = a_1^{i_1} a_2^{i_2} a_3^{i_3} a_4^{i_4} \dots$$

If $a_i^0 \neq a_i^1$ for infinitely many i it follows that $P(\{0, 1\}^\mathbb{N})$ contains uncountably many mutually tail inequivalent exit paths, contradicting our assumption on G . It follows therefore that there is an i_0 such that $a_i^0 = a_i^1$ for all $i \geq i_0$. This implies that z is tail-equivalent to $(v_i)_{i=1}^\infty$, and then by symmetry also to $(w_i)_{i=1}^\infty$. \square

Fix a vertex $v \in V$ and an increasing sequence $F_1 \subseteq F_2 \subseteq F_3 \subseteq \dots$ of finite subsets of V such that $v \in F_1$ and $\bigcup_n F_n = V$. Let $w = (w_i)_{i=1}^\infty$ be an exit path in G with $w_1 = v$ or an element $w \in V^n$ with $w_1 = v$ and $w_n \notin F_k$. There is then a unique vertex $E(w, k) \in V$ and a unique $j \in \mathbb{N}$ such that

- i) $w_j = E(w, k)$,
- ii) $w_i \notin F_k$, $i \geq j$,
- iii) $w_{j-1} \in F_k$.

In words, $E(w, k)$ is the first vertex outside of F_k which w visits when it leaves F_k for good.

Lemma 9.4. *Let G be a strongly connected graph with infinitely many vertexes and at most countably many exits. Then (\mathcal{E}, \leq) has a minimal element.*

Proof. This follows from Zorn's lemma if we prove that a maximal totally ordered subset M in (\mathcal{E}, \leq) has a minimal element. This is trivial if M is finite. By assumption M countable, say $M = \{t^i : i = 1, 2, 3, \dots\}$. For each $n \in \mathbb{N}$ we let s^n be the minimal element in $\{t^i : i = 1, 2, \dots, n\}$. Let $v^n = (v_i^n)_{i=1}^\infty$ be an exit path representing s^n such that $v_1^n = v$. Since G is row-finite there are infinite subsets $N_1 \supseteq N_2 \supseteq N_3 \supseteq \dots$ of natural numbers such that $E(v^n, k) = E(v^{n'}, k)$ for all $n, n' \in N_k$. We can therefore piece together an exit path $u = (u_i)_{i=1}^\infty$ such that for all $k \in \mathbb{N}$ there are infinitely many $n \in \mathbb{N}$ with $E(v^n, k) = E(u, k)$. It is easy to see that $(u_i)_{i=1}^\infty \leq (v_i^m)_{i=1}^\infty$ for all $m \in \mathbb{N}$. It follows that u represents an exit $[u]$ such that $[u] \leq t$ for all $t \in M$. \square

We can now finish the proof of Theorem 9.1 with the following

Lemma 9.5. *Let G be a strongly connected graph with infinitely many vertexes and at most countably many exits. A minimal element in (\mathcal{E}, \leq) is represented by a bare exit path.*

Proof. Let u be an exit path such that the exit $[u]$ it represents is minimal in (\mathcal{E}, \leq) . Assume for a contradiction that

$$\#s(r^{-1}(u_{i+1})) \geq 2 \tag{9.3}$$

for infinitely many i . There are then also infinite many $i \in \mathbb{N}$ such that (9.3) holds, and at the same time

$$k > i + 1 \Rightarrow u_k \neq u_{i+1}. \tag{9.4}$$

For each i for which (9.3) and (9.4) both hold we choose an edge $x_i \in r^{-1}(u_{i+1})$ with $s(x_i) \neq u_i$. Note that $\lim_{i \rightarrow \infty} s(x_i) = \infty$ since G is row-finite. Without loss of generality we can therefore assume that $s(x_i) \notin F_i$. For each i we choose a finite path μ_i in G with $s(\mu_i) = v$ such that x_i is the last edge in μ_i . Then $E(\mu_i, k)$ is defined for all $i \geq k$ and since G is row-finite there is a sequence $N_1 \supseteq N_2 \supseteq N_3 \supseteq \dots$ of infinite subsets in \mathbb{N} such that $E(\mu_i, k)$ is the same vertex for all $i \in N_k$. Let w_k be this common vertex. We can then piece together an exit path $y = (y_i)_{i=1}^\infty$ such that there is a sequence $l_1 \leq l_2 \leq \dots$ in \mathbb{N} with $y_{l_k} = w_k$ for all $k \in \mathbb{N}$. By construction there is then also a sequence $j_1 < j_2 < \dots$ in \mathbb{N} and for each $i \in \mathbb{N}$ a finite path μ'_i in G such that $s(\mu'_i) = y_{l_i}$, $\mu'_i \cap F_i = \emptyset$ and such that the last edge in μ'_i is x_{j_i} . Since the set N_i is infinite we can choose j_i arbitrarily large, for each l_i . The paths $\{\mu'_i\}$ ensure that $y \leq u$ and the minimality of $[u]$ implies then that y and u are tail-equivalent.

It follows that u has the following property: For each $i \in \mathbb{N}$ there is a finite path ν_i in G and natural numbers $l_i, k_i \in \mathbb{N}$ such that $\nu_i \cap F_i = \emptyset$, $l_i < k_i$, $s(\nu_i) = u_{l_i}$, $r(\nu_i) = u_{k_i}$, $j > k_i \Rightarrow u_j \neq u_{k_i}$, and the last edge in ν_i is not in $s^{-1}(u_{k_i-1})$. Furthermore, choosing the ν_i recursively we can arrange that $k_i < l_{i+1}$ for all i and

that ν_{i+1} does not contain a vertex which occurs in ν_j for some $j \leq i$. Set then $a_i^0 = u_{l_i} u_{l_i+1} \cdots u_{k_i}$ and let $a_i^1 \in \bigcup_n V^n$ be the string of vertexes in ν_i , starting with u_{l_i} and ending with u_{k_i} . Define $P : \{0, 1\}^\mathbb{N} \rightarrow P(V)$ such that

$$P((i_j)_{j=1}^\infty) = a_1^{i_1} u_{k_1+1} u_{k_1+2} \cdots u_{l_2-1} a_2^{i_2} u_{k_2+1} u_{k_2+2} \cdots u_{l_3-1} a_3^{i_3} u_{k_3+1} u_{k_3+2} \cdots u_{l_4-1} a_4^{i_4} \cdots$$

Note that the last occurrence in $P((i_j)_{j=1}^\infty)$ of each u_{k_i} and the vertex preceding it give away $(i_j)_{j=1}^\infty$. We conclude therefore that P is injective and that $P(\{0, 1\}^\mathbb{N})$ contains uncountably many tail-equivalence classes of exit paths in G , contradicting the assumption. This contradiction shows that u must be eventually bare. \square

REFERENCES

- [BR] O. Bratteli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics I + II*, Texts and Monographs in Physics, Springer Verlag, New York, Heidelberg, Berlin, 1979 and 1981.
- [BEK] O. Bratteli, G. Elliott and A. Kishimoto, *The temperature state space of a dynamical system I*, J. Yokohama Univ. **28** (1980), 125-167.
- [CL] T.M. Carlsen and N. Larsen, *Partial actions and KMS states on relative graph C^* -algebras*, Preprint, Nov. 2013.
- [CT1] J. Christensen and K. Thomsen, *Finite digraphs and KMS states*, J. Math. Anal. Appl. **433** (2016), 1626-1646.w
- [CT2] J. Christensen and K. Thomsen, *Diagonality of actions and KMS weights*, preprint, arXiv:1512.05086.
- [Co] D.L. Cohn, *Measure Theory*, Birkhäuser, Boston, Basel, Berlin, 1997.
- [C] F. Combes, *Poids associé à une algèbre hilbertienne à gauche*, Compos. Math. **23** (1971), 49-77.
- [Cy] V.T. Cyr, *Transient Markov Shifts*, Ph.D. thesis, Pennsylvania State University, August 2010.
- [EFW] M. Enomoto, M. Fujii and Y. Watatani, *KMS states for gauge action on O_A* , Math. Japon. **29** (1984), 607-619.
- [EL] R. Exel and M. Laca, *Partial dynamical systems and the KMS condition*, Comm. Math. Phys. **232** (2003), 223-277.
- [Ki] A. Kishimoto, *Locally representable one-parameter automorphism groups of AF algebras*, Rep. Math. Phys. **45** (2000), 333-356.
- [KV] J. Kustermans and S. Vaes, *Locally compact quantum groups*, Ann. Scient. Éc. Norm. Sup. **33**, 2000, 837-934.
- [LN] M. Laca and S. Neshveyev, *KMS states of quasi-free dynamics on Pimsner algebras*, J. Func. Analysis **211** (2004), 457-482.
- [N] S. Neshveyev, *KMS states on the C^* -algebras of non-principal groupoids*, J. Operator Theory **70** (2013), 513-530.
- [Pa] A.L.T. Paterson, *Graph inverse semigroups, groupoids and their C^* -algebras*, J. Operator Theory **48** (2002), 645-662.
- [Ph] N. C. Phillips, *Crossed Products of the Cantor Set by Free Minimal Actions of \mathbb{Z}^d* , Comm. Math. Phys. **256** (2005), 1-42.
- [Pr] W.E. Pruitt, *Eigenvalues of non-negative matrices*, Ann. Math. Statist. **35** (1964), 1797-1800.
- [Re] J. Renault, *A Groupoid Approach to C^* -algebras*, LNM 793, Springer Verlag, Berlin, Heidelberg, New York, 1980.
- [Ru] S. Ruelle, *On the Vere-Jones classification and existence of maximal measure for countable topological Markov chains*, Pac. J. Math. **209** (2003), 365-380.
- [S] O. Sarig, *Thermodynamic formalism for countable Markov shifts*, Ergodic Th. & Dynam. Syst. **19** (1999), 1565-1593.

- [Sa] S.A. Sawyer, *Martin boundary and random walks*, Harmonic functions on trees and buildings (New York, 1995), 17-44, Contemp. Math. 206, Amer. Math. Soc., Providence, RI, 1997.
- [Sz] W. Szymanski, *Simplicity of Cuntz-Krieger algebras of infinite matrices*, Pac. J. Math. **122** (2001), 249-256.
- [Th1] K. Thomsen, *KMS weights on groupoid and graph C^* -algebras*, J. Func. Analysis **266** (2014), 2959-2988.
- [Th2] K. Thomsen, *Dissipative conformal measures on locally compact spaces*, Ergod. Th. & Dynam. Syst. **36** (2016), 649-670.
- [Th3] K. Thomsen, *On the positive eigenvalues and eigenvectors of a non-negative matrix*, to appear in the Abel Symposium 2015 Proceedings. arXiv:1306.5116.
- [V] D. Vere-Jones, *Ergodic properties of non-negative matrices I*, Pacific J. Math. **22** (1967), 361-386.
- [Wo] W. Woess, *Denumerable Markov Chains*, EMS Textbooks in Mathematics, 2009.

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